On blowup dynamics in the Keller-Segel model of chemotaxis

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In memory of V.S. Buslaev, a scientist and a friend
Will appear in St.Petersburg Math Journal (issue dedicated to V.S. Buslaev)"

Abstract

We investigate the (reduced) Keller-Segel equations modeling chemotaxis of bio-organisms. We present a formal derivation and partial rigorous results of the blowup dynamics of solution of these equations describing the chemotactic aggregation of the organisms. Our results are confirmed by numerical simulations and the formula we derive coincides with the formula of Herrero and Velázquez for specially constructed solutions.

1 Introduction

In this paper we analyze the aggregation dynamics in the (reduced) Keller-Segel model of chemotaxis. Chemotaxis is the directed movement of organisms in response to the concentration gradient of an external chemical signal and is common in biology. The chemical signals can come from external sources or they can be secreted by the organisms themselves.

Chemotaxis is believed to underly many social activities of micro-organisms, e.g. social motility, fruiting body development, quorum sensing and biofilm formation. A classical example is the dynamics and the aggregation of *Escherichia coli* colonies under starvation conditions [17]. Another example is the *Dictyostelium* amoeba , where single cell bacterivores, when challenged by adverse conditions, form multicellular structures of $\sim 10^5$ cells [15, 23]. Also, endothelial cells of humans react to vascular endothelial growth factor to form blood vessels through aggregation [22].

Consider organisms moving and interacting in a domain $\Omega \subseteq \mathbb{R}^d$, $d = 1, 2$ or 3. Assuming that the organism population is large and the individuals are small relative to the domain Ω, Keller and Segel derived a system of reaction-diffusion equations governing the organism density $\rho : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ and chemical concentration $c : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$. The equations are of the form

$$\begin{aligned} \partial_t \rho &= D_\rho \Delta\rho - \nabla \cdot \big(f(\rho)\nabla c\big) \\ \partial_t c &= D_c \Delta c + \alpha\rho - \beta c. \end{aligned} \tag{1}$$

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Here D_ρ, D_c, α, β are positive functions of x and t, ρ and c, and $f(\rho)$ is a positive function modeling chemotaxis. Assuming a closed system, one is led to impose no-flux boundary conditions on ρ and c:

$$\partial_\nu \rho = 0 \text{ and } \partial_\nu c = 0 \text{ on } \partial\Omega, \tag{2}$$

where $\partial_\nu g$ is the normal derivative of g. With these boundary conditions, the total number of organisms in Ω is conserved. We refer the reader to [15, 17, 50, 58] for more information about chemotaxis and the Keller-Segel model.

We presently concentrate on the case of positive chemotaxis, where the organisms secrete the chemical and move towards areas of higher chemical concentration. This leads to aggregation of organisms. Mathematically this is expressed as a blowup (or collapse) of solutions of (1). It was first suggested by Nanjundiah in [58] that the density, ρ, may become infinite and form a Dirac delta singularity. One refers to this process as (chemotactic) collapse. This is, arguably, the most interesting feature of the Keller-Segel equations. As argued below, the "collapsing" profile and contraction law have a universal (close to self-similar) form, independent of particulars of initial configurations and, to a certain degree, of the equations themselves, and can be associated with chemotactic aggregation. Though the equations are rather crude and unlikely to produce patterns one observes in nature or experiments, the collapse phenomenon could be useful in verifying assumptions about biological mechanisms.[1]

Phenomena of blowup and collapse in nonlinear evolution equations are hard to simulate numerically and the rigorous theory, or at least a careful analysis, is pertinent here. The recent years witnessed a tremendous progress in the development of such theories. We can now describe the shape of blowup profile and contraction law in Yang-Mills, $\sigma-$model, nonlinear Schrödinger and heat equations ([75, 73, 51, 52, 70, 8, 56, 57, 29]) [2]. Yet, after 40 years of intensive research and important progress, we still cannot give a rigorous description of collapse in the Keller-Segel equations modeling chemotaxis. (See [17, 86, 87, 10, 11, 12, 13, 14, 4] for some recent works, [16], for a nice discussion of the subject, and [60, 45, 46, 42, 72] for reviews.)

This is not to say that the Keller-Segel equations are harder than Yang-Mills, $\sigma-$model, or nonlinear Schrödinger equations, they are not, but neither are they less important.

There are three common approximations made in the literature for system (1). Firstly, one assumes that the coefficients in (1) are constant and satisfy

$$\epsilon := \frac{D_\rho}{D_c} \ll 1,\ \tilde{\alpha} := \frac{\alpha}{D_c} = \mathrm{O}\big(1\big) \text{ and } \tilde{\beta} := \frac{\beta}{D_c} \ll 1. \tag{3}$$

The first of these conditions states that the chemical diffuses much faster than the organisms do. This is the case in practically all situations. As a result of this relation, one drops the $\partial_t c$ term in (1) (after rescaling time $t \to t/D_\rho$, this term becomes $\epsilon\partial_t c$). Secondly, one takes $f(\rho)$ to be a linear function $f(\rho) = K\rho$. Thirdly, the term βc in (1) is neglected compared with $\alpha\rho$, as one expects that it would not effect the blow-up process where $\rho \gg 1$ (it is also small due to the last relation in (3)). These approximations, after rescaling, lead to the system

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= \Delta\rho - \nabla\cdot\big(\rho\nabla c\big), \\ 0 &= \Delta c + \rho, \end{aligned} \tag{4}$$

[1]There are numerous refinements of the Keller-Segel equations, e.g. taking into account finite size of organisms ([1, 2, 55]) preventing complete collapse, which model chemotaxis more precisely. We believe the techniques we outline and develop here can be applied to these models as well.

[2]Numerical simulations for these equations failed until the compression rate was derived analytically, see [8, 70, 82]

with ρ and c satisfying the no-flux Neumann boundary conditions.

Equations (4) in three dimensions also appear in the context of stellar collapse (see [40, 88, 24, 76]); similar equations—the Smoluchowski or nonlinear Fokker-Planck equations—models non-Newtonian complex fluids (see [31, 53, 26, 27]. This is the equation studied in this paper.

We emphasize that in dropping the time derivative term of c, we have made the adiabatic approximation, in which the chemical is assumed to reach its steady state given by the second equation of (4) instantaneously.

In this paper, we consider the collapse of radially symmetric solutions to the reduced Keller-Segel system (4) on the plane $\mathbb{R}^2$ with a smooth, positive and integrable initial condition ρ_0 and with the boundary conditions $\rho, \nabla\rho, \nabla c \to 0$ as $|x| \to \infty$. To provide a right context for the discussion below, we mention that equation (4) has the following key properties:

- It is invariant under the scaling transformations

$$\rho(x,t) \to \frac{1}{\lambda^2}\rho\big(\frac{1}{\lambda}x, \frac{1}{\lambda^2}t\big) \text{ and } c(x,t) \to c\big(\frac{1}{\lambda}x, \frac{1}{\lambda^2}t\big). \tag{5}$$

- It has the static solution,

$$R(x) := \frac{8}{(1+|x|^2)^2},\; C(x) := -2\ln(1+|x|^2). \tag{6}$$

- The total "mass" is conserved: $\int_\Omega \rho(x,t)\,dx = \int_\Omega \rho(x,0)\,dx$.

We also mention that (4) (as well as (1)) is a gradient flow, $\partial_t\rho = \nabla\cdot\rho\nabla\mathcal{E}'(\rho)$, or $\partial_t\rho = -\mathrm{grad}\,\mathcal{E}(\rho)$, where $\mathcal{E}'(\rho)$ is the formal L^2-gradient of $\mathcal{E}$ and $\mathrm{grad}\,\mathcal{E}(\rho)$ is the formal gradient of $\mathcal{E}$ in the space with metric $\langle v,w\rangle_J := -\langle v, J^{-1}w\rangle_{L^2}$. Here $J := \nabla\cdot\rho\nabla \le 0$, whose inverse is unbounded operator, and $\mathcal{E}(\rho)$ is the "energy" functional given by

$$\mathcal{E}(\rho) = \int_{\mathbb{R}^2}(-\frac{1}{2}\rho\Delta^{-1}\rho + \rho\ln\rho - \rho)\,dx \tag{7}$$

(see Appendix C for more details). We remark that the first term of $\mathcal{E}$ can be thought of as the internal energy of the system and the remaining terms are the entropy. The solution (6) is a minimizer of $\mathcal{E}$ under the constraint that $\int\rho = \text{const}$. Note that $\int_{\mathbb{R}^2} R\,dx = 8\pi$, which is the source of 8π in (8). Under the scaling (5), the total mass changes as

$$\int \frac{1}{\lambda^2}\rho\big(\frac{1}{\lambda}x, 0\big) = \lambda^{(d-2)}\int \rho\big(x,0\big).$$

Thus one does not expect collapse for $d=1$, and that collapse is possible for $d \ge 2$ with critical collapse for $d=2$ and supercritical collapse for $d>2$. (Equation (4) in $d=2$ is said to be L^1-critical, etc.)

Take $\rho_0 \ge 0$. One has the following criteria for blowup of solutions of (4) ([59, 6]): If the dimension $d=2$ and the total mass satisfies

$$M := \int_{\mathbb{R}^2}\rho_0\,dx > 8\pi, \tag{8}$$

or, if $d \ge 3$ and $\frac{\int_{\mathbb{R}^d} x^2\rho_0 dx}{\int_{\mathbb{R}^d}\rho_0 dx}$ is sufficiently small (this means that ρ_0 is concentrated at $x=0$), then the solution to (4) blows up in finite time.

There is a fair amount of work done on equations (1) and (4) and closely related equations. We give a very brief and incomplete review of it. Childress and Percuss [25] found that collapse for (1) with f linear does not occur when $d = 1$ and can occur when $d \geq 3$. For the two-dimensional case, they advanced arguments that collapse requires a threshold number of organisms. This threshold behaviour was confirmed by Jäger and Luckhaus in [49] (see also [59, 63, 62, 65, 65, 64, 60]).

Herrero and Velázquez proved that there exist radial solutions of (4) for $d = 2$ with the threshold mass $8|\Omega|$ collapsing to a Dirac delta singularity in finite time (see [37]). Also, unlike previous results, the authors give an explicit asymptotic expression of the developing singularity. They proved using matched asymptotics and a topological argument that for $T > 0$ there exists a radial solution to (4), which blows up at $r = 0$ and $t = T$ and is of the form

$$\rho(r,t) = \frac{1}{\lambda(t)^2} R_{\lambda(t)}(1 + \mathrm{o}(1)) + \begin{cases} 0 & r < \lambda(t) \\ \mathrm{O}\big(\frac{e^{-\sqrt{2}|\ln(T-t)|^{\frac{1}{2}}}}{r^2}\big) & r \geq \lambda(t), \end{cases}$$

as $t \to T$, where $R_\lambda(r) := R(r/\lambda)$, $R(r)$ is the stationary solution to (4) (see (6)) and

$$\lambda(t) = C(T-t)^{\frac{1}{2}} e^{-\frac{1}{\sqrt{2}}|\ln(T-t)|^{\frac{1}{2}}} |\ln(T-t)|^{\frac{1}{4}|\ln(T-t)|^{-\frac{1}{2}} - \frac{1}{4}} (1 + \mathrm{o}(1)).$$

They also considered collapse of solutions to (1) with linear $f(\rho)$ (see [38] and [39]). Obtaining similar results, they suggest that Jäger and Luckhaus' adiabatic assumption does not affect the collapse mechanism. In the papers [54, 32] Lushnikov et al derived the log-log scaling as well as corrections beyond leading order log-log scaling.

As noted in [37], the asymptotics reproduced above are not of self-similiar type; that is, they are not of the form $(T-t)\Phi(r/(T-t)^{\frac{1}{2}})$ for some function Φ. In fact, as shown in [41], self-similiar blowup is not possible. Lastly, we mention that similar work has been done for the three dimensional case, where existence of collapsing shock waves has been shown. We refer the reader to [41, 40, 16] for these results.

The above results are valid for radially symmetric domains and initial conditions. It was shown by numerous authors that the blowup threshold mentioned above decreases for the non-spherically symmetric situation. Moreover, Dirac delta singularities may develop on the boundary of the domain. We refer the reader to [5, 6, 33, 43, 47, 61] for details. In [86], Velázquez considers small radial and non radial perturbations of a collapsing solution and concludes, using formal matched asymptotics, that they are stable to these perturbations, leading only to small shifts in the blowup time and the blowup point. Existence of blowup or bounded solutions when $f(\rho)$ is nonlinear was recently studied in [48]. We also refer to [44] for a blowup result of a related Keller-Segel model. Lastly, we refer the reader to Horstmann [45, 46] for a more complete review of the literature including results on other models of chemotaxis and on the derivation of the Keller-Segel model as a continuous limit of biased random walks (see e.g. [67, 69, 80]).

In spite of the considerable progress, the question of whether the mass collects in isolated points, forming Dirac delta distributions, remained unanswered. Moreover, these results give no information about the dynamics of blowup. These are the questions we address.

Now, we describe the results of the present paper. Given a radially symmetric initial condition $\rho_0(r) > 0$ sufficiently close to some R_{λ_0}, for some λ_0, and satisfying $\int \rho_0 > \int R$, we show (formally, but with some rigorous supporting results) that the solution $\rho(x,t)$ to (4) is of the form

$$\rho(x,t) = \frac{1}{\lambda^2(t)} R_{\lambda(t)}(r)(1 + \mathrm{o}(1))$$

with $\lambda(t) \to 0$ as $t \to T$ for some $0 < T < \infty$. Thus, all the mass $\int \rho\, dx$ collapses to the single point $x = 0$ in finite time, or equivalently, the density ρ forms a Dirac delta singularity with weight 8π in finite time. Furthermore, we show that the compression scale, λ, has the following explicit asymptotics

$$\lambda(t) = c(T-t)^{\frac{1}{2}} e^{-\frac{1}{\sqrt{2}}|\ln(T-t)|^{\frac{1}{2}}} |\ln(T-t)|^{\frac{1}{4}}(1 + \mathrm{o}(1)) \tag{9}$$

for some constant c. In Figure 1 we compare the blowup asymptotics (9) with direct numerical simulation of (4).

We also give an estimate of the error term, $\rho(x,t) - R_{\lambda(t)}(r)$, in the case when the nonlinear part in equation (16), given below, can be neglected. We believe that our results and our analysis can be made rigorous and can be extended to the full Keller-Segel system.

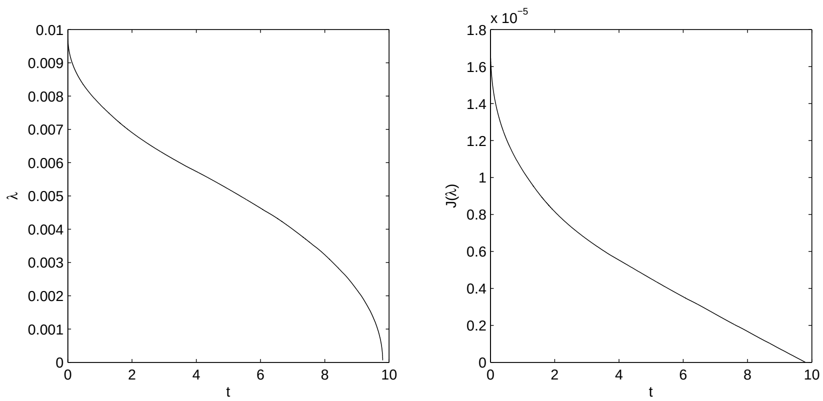

Figure 1: The left pane shows the scaling parameter $\lambda(t)$ obtained by numerically computing the solution to (4) with the initial condition $m_0 := 4y^2/(1 + \delta y + y^2)|_{y=r/\lambda_0}$ with $\lambda_0 = -\delta = .01$. The right pane plots the quantity $J(\lambda) := \dfrac{e^{\sqrt{4 \ln \frac{\lambda_0}{\lambda}}}}{\sqrt{\ln \frac{\lambda_0}{\lambda}}} (\frac{\lambda}{\lambda_0})^2$ against time, which according to (9) should be linear as the blowup time is approached.

We outline the approach used in this paper. In the case of radially symmetric solutions, the system (4), which consists of coupled parabolic and elliptic PDEs, is equivalent to a single PDE. Indeed, the change of the unknown, by passing from the density, $\rho(x,t)$, to the normalized mass,

$$m(r,t) := \frac{1}{2\pi} \int_{|x| \le r} \rho(x,t)\ dx,$$

of organisms contained in a ball of radius r, discovered by [49, 16], maps two equations (4) into a single equation

$$\partial_t m = \Delta_r^{(0)} m + r^{-1} m \partial_r m, \tag{10}$$

on $(0, \infty)$ (with initial condition $m_0(r) := \frac{1}{2\pi} \int_{|x| \le r} \rho_0(x)\, dx$). Here $\Delta_r^{(n)}$ is the n-dimensional radial Laplacian, $\Delta_r^{(n)} := r^{-(n-1)} \partial_r r^{n-1} \partial_r = \partial_r^2 + \frac{n-1}{r} \partial_r$. Thus (4) in the radially symmetric case is equivalent to (10) and therefore we concentrate on the latter equation.

The properties of equation (4) discussed above imply the following key properties equation of (10)

- It is invariant under the scaling transformations $m(r,t) \to m\big(\frac{1}{\lambda}r, \frac{1}{\lambda^2}t\big)$.
- It has the static solution (coming from the static solution $R(r) = \frac{8}{(1+r^2)^2}$ of (4)),

$$\chi(r) := \frac{4r^2}{1+r^2}. \tag{11}$$

- The total "mass" is conserved: $2\pi \lim_{r\to\infty} m(r,t) = \int \rho(x,t)dx = \text{const}$.

Note that the stationary solution has total mass $2\pi \lim_{r\to\infty} \chi(r) = 8\pi$, which, recall, is the sharp threshold between global existence and singularity development in solutions to (4) (see (8)).

The properties above yield, as in the case of (4), the manifold of static solutions $\mathcal{M}_0 := \{\chi(r/\lambda) \mid \lambda > 0\}$ and suggest a likely scenario of collapse: sliding along $\mathcal{M}_0$ in the direction of $\lambda \to 0$. To analyze the collapse, we pass to the reference frame collapsing with the solution, by introducing the adaptive blowup variables,

$$m(r,t) = u(y,\tau), \qquad \text{where} \qquad y = \frac{r}{\lambda} \text{ and } \tau = \int_0^t \frac{1}{\lambda^2(s)}\,ds,$$

where $\lambda : [0,T) \to [0,\infty)$, $T > 0$, is a positive differentiable function (*compression* or *dilatation* parameter), such that $\lambda(t) \to 0$ as $t \uparrow T$. The advantage of passing to blowup variables is that the function u is expected to have bounded derivatives and the blowup time is eliminated from consideration (it is mapped to ∞). Writing (10) in blowup variables, we find the equation for the rescaled mass function

$$\partial_\tau u = \Delta_y^{(0)} u + y^{-1} u_\lambda \partial_y u - ay\partial_y u, \tag{12}$$

where $a := -\dot{\lambda}\lambda$. Now, the blowup problem for (10) is mapped into the problem of asymptotic dynamics of solitons for the equation (12), which was already studied in the pioneering works of [77, 78, 79, 18, 19, 83, 84, 85, 34, 35].

The boundary conditions on u are $\partial_y^\alpha u(y,\tau) \to 0$ as $y \to \infty$ for $\alpha = 1, 2$. As with the boundary conditions for (10), these imply that mass is conserved: $\lim_{y\to\infty} u(y,\tau) = \lim_{y\to\infty} u(y,0)$. Equivalently, u, as a solution of (12), depends on a, which determines λ, given $\lambda(0) = \lambda_0$, according to the formula

$$\lambda^2(t) = \lambda_0^2 - 2\int_0^t a(s)\,ds. \tag{13}$$

Equation (12) has the static solution $(\chi(y), a = 0)$. It is shown in [30] that the linearized operator on this solution has one negative eigenvalue $-2a + \frac{a}{\ln\frac{1}{a}} + O\big(a \ln^{-2}\frac{1}{a}\big)$ (corresponding to the scaling mode—for a fixed parabolic scaling it is connected to possible variation of the blowup time) [3] and one near zero eigenvalue, while the third eigenvalue, $2a + \frac{2a}{\ln\frac{1}{a}} + O\big(a \ln^{-2}\frac{1}{a}\big)$, is positive, but vanishing as $a \to 0$. (It also isolates the correct perturbation (adiabatic) parameter—$\frac{1}{\ln\frac{1}{a}}$.) Hence we have to construct a one-parameter deformation of $\chi(y)$ (besides the parameter λ, or a). For technical reasons it is convenient to use a two-parameter family, $\chi_{bc}(y)$

$$\chi_{bc}(y) := \frac{4by^2}{c+y^2}, \tag{14}$$

[3]A similar analysis applies also in the subcritical case $M < 8\pi$ where the solution converges to a self-similar one as $\tau \to \infty$, which vanishes as $t \to \infty$. In this case the operator $\mathcal{L}_{ab}$ has strictly positive spectrum.

with $b > 1$ and both parameters b and c are close to 1, with an extra relation between the parameters a, b and c. The family $\chi_{bc}(y)$ gives approximate solutions to (12) (see (43)) and forms the deformation (or almost center-unstable) manifold $\mathcal{M} := \{\chi_{bc}(r/\lambda) \mid \lambda > 0,\ p\}$. We expect that the solution to (12) approaches this manifold as $\tau \to \infty$, and therefore we decompose the solution $u(y,\tau)$ to (12) as the leading term, $\chi_{b(\tau)c(\tau)}(y)$, and the fluctuation, $\phi(y,\tau)$,

$$u(y,\tau) = \chi_{b(\tau)c(\tau)}(y) + \phi(y,\tau), \tag{15}$$

and require that the fluctuation $\phi(y,\tau)$ is orthogonal to the tangent space of $\mathcal{M}$ at $\chi_{b(\tau)c(\tau)}(y)$, $\langle \partial_p \chi_{p(\tau)}(\cdot), \phi(\cdot,\tau)\rangle = 0$, where $p := (b,c)$. Note that this family evolves on a different spatial scale than $\phi(y,\tau)$ in (15), as it can rewritten as $\chi_{bc}(y) = \chi_{\frac{b}{c},1}(\frac{y}{\sqrt{c}}) = \chi_{bc}$.

In parametrizing solutions as above, we split the dynamics of (4) into a finite-dimensional part describing motion over the manifold, $\mathcal{M}$, and an infinite-dimensional fluctuation (the error between the solution and the manifold approximation) which is supposed to stay small. Substituting the decomposition (15) into the equation (12), we arrive at the equation

$$\partial_\tau \phi = -\mathcal{L}_{abc}\phi + \mathcal{F}_{abc} + \mathcal{N}(\phi), \tag{16}$$

where $\mathcal{L}$ is a self-adjoint linear operator, $\mathcal{F}$ is a forcing term, and $\mathcal{N}$ is a quadratic nonlinearity. Due to the definition of $\mathcal{M}$, it turns out that its tangent space is very close to the subspace spanned by the negative and almost zero spectrum eigenfunctions (unstable modes) of the linearized operator, $\mathcal{L}_{abc}$, and therefore ϕ is (approximately) orthogonal to the latter subspace.

The contraction law is obtained by using the orthogonality condition, $\langle \partial_{bc}\chi_{bc}, \phi\rangle = 0$. The latter is equivalent to two conditions,

$$\partial_\tau \langle \partial_{bc}\chi_{b(\tau)c(\tau)}(\cdot), \phi(\cdot,\tau)\rangle = 0 \tag{17}$$

and $\langle \partial_{bc}\chi_{b(\tau)c(\tau)}(\cdot), \phi(\cdot,\tau)\rangle|_{t=0} = 0$, which lead, to leading order, to the differential equation

$$a_\tau = -\frac{2a^2}{\ln(\frac{1}{a})}, \tag{18}$$

whose solutions, to leading order, are (9) (see Section 4).

We now describe the organization of this paper. In Section 2, solutions to (4) are parametrized by the parameters (a,b,c,ϕ) connected to u by (12). In Section 3, we study the operator $\mathcal{L}_{abc}$ in (16) and show that it has one negative eigenvalue and one simple eigenvalue near zero. We also give approximate eigenfunctions corresponding to these eigenvalues and prove that $\mathcal{L}_{abc}$ is positive on the space orthogonal to these quasi-eigenfunctions. In Section 4, we state the relationship between the blowup parameters a, b and c, whose proof is given in Appendix D, and use it to obtain a dynamical equation for the blowup parameter $a = -\lambda\partial_t\lambda$ and derive the leading order behaviour of the scaling parameter λ in terms of the original time variable. In Section 5, we derive the lower bounds on the operator $\mathcal{L}_{abc}$. We use these bounds in in Section 6 in order to control the fluctuation ϕ in the linearized equation, i.e. for (16), with the nonlinearity $\mathcal{N}(\phi)$ omitted.

In Appendix A we present the family of solutions to (10),

$$\chi^{(\mu)}(r) := \frac{r^{\mu-2}\mu + 4 - \mu}{r^{\mu-2} + 1}$$

with mass $2\pi\mu$, where $\mu \in (2,4]$. These solutions describe partial collapse with $2\pi(4-\mu)$ units of mass concentrated at the origin. In the remainder of our work we will make no further use of these partially

collapsed solutions. In Appendix B we provide a proof of the orthogonal splitting theorem of Section 2 and in Appendix C we discuss te gradient structure of equations (1) and (4).

In the following discussion, we use the notation $f \lesssim g$ if there exists a positive constant C such that $f \leq Cg$ holds. If the inequality $|f| \leq C|g|$ holds then we write $f = \mathrm{O}(g)$. We also write $f \ll g$ or $f = \mathrm{o}(g)$ if $f(a)/g(a) \to 0$ as $a \to 0$ and $f \sim g$ if the quotient converges to 1.

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2 Parametrization of Solutions

We parameterize solutions $u_\lambda(y,\tau)$ of equation (12) by the parameters $a,\ b$ and c, and the fluctuation ϕ according to

$$u_\lambda(y,\tau) = \chi_{bc}(y) + \phi(y,\tau) \tag{19}$$

where $\lambda = \lambda(\tau),\ b = b(\tau)$ and $a = a(\tau)$. Substituting decomposition (19) into equation (12) gives that the fluctuation ϕ satisfies

$$\partial_\tau \phi = -\mathcal{L}_{abc}\phi + \mathcal{F}_{abc} + \mathcal{N}(\phi), \tag{20}$$

where the linear operator, the forcing terms, and the nonlinear term are

$$\mathcal{L}_{abc} := -\Delta^{(4)} - \frac{8bc}{(c+y^2)^2} - \frac{4}{y}(b-1-\frac{bc}{c+y^2})\partial_y + ay\partial_y, \tag{21}$$

$$\mathcal{F}_{abc} := -4b_\tau + 4\frac{b_\tau c + bc_\tau - 2bca}{c+y^2} + 4bc\frac{8(b-1)+2ac-c_\tau}{(c+y^2)^2} - \frac{32bc^2(b-1)}{(c+y^2)^3}, \tag{22}$$

$$\mathcal{N}(\phi) := y^{-1}\phi\partial_y\phi. \tag{23}$$

Consider the weighted L^2-space $L^2(\mathbb{R}_+, \gamma_{ab}(y)y^3dy)$, with the weight

$$\gamma_{abc}^{-1/2}(y) = \frac{4y^2 e^{\frac{a}{4}y^2}}{(c+y^2)^b}, \tag{24}$$

and the corresponding inner product

$$\langle f, g\rangle := \int_0^\infty f(y)g(y)\,\gamma_{abc}(y)y^3dy. \tag{25}$$

The norm corresponding to this inner will be denoted by $\|\cdot\|$. The significance of this space is that, as we show below, the operator $\mathcal{L}_{abc}$ is self-adjoint on it.

Remark. Another way to write $\mathcal{L}_{abc}$ is as

$$\mathcal{L}_{abc} = -\Delta^{(0)} - \frac{8bc}{(c+y^2)^2} - \frac{4by}{c+y^2}\partial_y + ay\partial_y, \tag{26}$$

and treat it as a self-adjoint operator on $L^2(\mathbb{R}_+, \tilde{\gamma}_{ab}(y)y^3dy)$, with weight $\tilde{\gamma}_{abc}^{-1/2}(y) = \frac{e^{\frac{a}{4}y^2}}{(c+y^2)^b}$, and corresponding inner product $\langle f, g\rangle := \int_0^\infty f(y)g(y)\,\tilde{\gamma}_{abc}(y)y^{-1}dy$.

The decomposition (19) is not unique and as a result we have a single equation, (20), for four unknowns, a, b, c and ϕ. Hence we supplement equation (20) with three additional equations. Two of the equations can be chosen as in [29] to make the parameters a, b, and c satisfy a chosen relation, say $f(a,b,c)=0$. In addition, we have the relations

$$\langle \phi, \zeta_{bci} \rangle = 0,\ i = 0,1, \tag{27}$$

in $L^2(\mathbb{R}^+, \gamma_{abc}(y)y^3dy)$, for all times $\tau > 0$, where ζ_{bci} are the tangent vectors to the manifold $\mathcal{M} := \{\chi_{bc}(r/\lambda) \mid \lambda > 0,\ b,\ c\}$:

$$\zeta_{bc0}(y) := \frac{1}{8bc} y\partial_y \chi_{bc}(y) = \frac{y^2}{(c+y^2)^2}, \qquad \zeta_{bc1}(y) := \frac{1}{4}\partial_b \chi_{bc}(y) = \frac{y^2}{c+y^2}, \tag{28}$$
$$\zeta_{bc2}(y) := -\frac{1}{4b}\partial_c \chi_{bc}(y) = \frac{y^2}{(c+y^2)^2}.$$

(The vectors $\zeta_{bc0}(y)$ and $\zeta_{bc2}(y)$ are seen to be multiples of each other which confirms that one of the parameters is superfluous.)

We proceed here differently and choose

$$-\langle \phi, \partial_\tau \zeta_{bci} + (\partial_\tau \ln \gamma_{ab})\zeta_{bci} \rangle = -\langle \mathcal{L}_{abc}\phi, \zeta_{bci} \rangle + \langle \mathcal{F}_{abc}, \zeta_{bci} \rangle + \langle \mathcal{N}, \zeta_{bci} \rangle,\ i = 0,1. \tag{29}$$

As we will show the latter vectors are approximate eigenvectors of the operator $\mathcal{L}_{abc}$ having the negative and almost zero eigenvalues. In addition, we will choose a relation between the parameters a, b, c. Eqns (29) imply that

$$\partial_\tau \langle \phi, \zeta_{bci} \rangle = 0\,, \tag{30}$$

and therefore the inner products $\langle \phi, \zeta_{bci} \rangle$, $i = 0,1$, are constant (one can think of this a constraint on a, b and c). The next proposition shows that a_0, b_0 and c_0 can be taken so that $\langle \zeta_{bci}, \phi \rangle|_{\tau=0} = 0$, and hence, by (29), we have (27). To be able to formulate a precise statement we introduce, for a fixed $\delta > 0$, open neighbourhoods of $\mathcal{M}$,

$$\mathcal{U}_\varepsilon = \{f : \|\mathrm{e}^{-\frac{\delta}{3}y^2}\big(f(y) - \chi_{bc}(y)\big)\|_\infty < \varepsilon\,, \text{for some } 1 \le b \le 2, \frac{1}{2} \le c \le 1\} \tag{31}$$

and, for a fixed $\lambda > 0$,

$$\tilde{\mathcal{U}}_\varepsilon = \{f(r) : f(\frac{r}{\lambda}) \in \mathcal{U}_\varepsilon\}\,. \tag{32}$$

Proposition 2.1. *Fix $\lambda_0 > 0$ and $0 < \delta \ll 1$. Then there is an $\varepsilon > 0$ and a unique C^1 function $g : \mathcal{U}_\varepsilon \to (\delta, 1) \times (1/2, 1)$ such that for $m_0 \in \tilde{\mathcal{U}}_\varepsilon$ we have the equation $\langle \zeta_{b_0c_0i}, \phi_0 \rangle = 0$, or in detail, for $i = 1,2$,*

$$\langle m_0(\lambda_0 \cdot) - \chi_{b_0c_0}, \zeta_{b_0c_0i} \rangle|_{(a_0,c_0)=g(m_0)} = 0. \tag{33}$$

For the proof of this proposition see Appendix B.

Equations (20) and (29) form a system of coupled, partial and ordinary differential equations for the parameters ϕ, a, b, and c. We assume that this system has a unique local solution given initial conditions ϕ_0, a_0, b_0 and c_0, the values of which are related to the initial value of m (recall that $u_\lambda(y) = m(\lambda y)$).

3 General Properties of the Operator $\mathcal{L}_{abc}$

Before proceeding we discuss general properties of the operator $\mathcal{L}_{abc}$ mentioned above and used below,

Proposition 3.1. *The operator $\mathcal{L}_{abc}$, defined on $L^2([0,\infty),\gamma_{abc}(y)y^3dy)$ (with inner product (25)), is self-adjoint and has purely discrete spectrum. Moreover, we have the lower bound*

$$\langle\phi,\mathcal{L}_{abc}\phi\rangle \gtrsim -[2a+\frac{(b-1)+a}{\sqrt{\ln\frac{1}{a}}}]\|\phi\|^2\,. \tag{34}$$

Proof. One can check the self-adjointness of $\mathcal{L}_{abc}$ directly or use the unitary map $\xi(y)\to\gamma_{abc}^{1/2}(y)\xi(y)$, from $L^2([0,\infty),\gamma_{abc}(y)y^3dy)$ to $L^2([0,\infty),y^3dy)$ to map this operator into the operator

$$L_{abc}:=\gamma_{abc}^{1/2}\mathcal{L}_{abc}\gamma_{abc}^{-1/2}, \tag{35}$$

acting on $L^2([0,\infty),y^3dy)$ with inner product $(\xi,\eta):=\int\xi\eta y^3dy$. The latter operator can be explicitly computed to be

$$L_{abc}:=-\Delta^{(4)}+\frac{1}{4}a^2y^2-2ab+\frac{2b\big(2(b-1)+ac\big)}{c+y^2}-\frac{4bc(b+1)}{(c+y^2)^2}\,. \tag{36}$$

It is of Schrödinger type with the real continuous potential tending to ∞ as $y\to\infty$ as $O(y^2)$. Hence, using standard arguments (see e.g. [36]), one can show that L_{abc} is self-adjoint and its spectrum, and hence the spectrum of $\mathcal{L}_{abc}$, is purely discrete.

Now, we investigate the bottom of the spectrum of the operator $\mathcal{L}_{abc}$. We begin with the operator $\mathcal{L}_{0bc}:=\mathcal{L}_{abc}|_{a=0}$. In what follows we use the convenient shorthand notation $f_\lambda(r)=f(r/\lambda)$, which we apply only for the subscript λ.

Lemma 3.2. *The function $\zeta_{bc}(y):=\frac{y^2}{(c+y^2)^{2b}}$ is a zero mode of the operator $\mathcal{L}_{0bc}:=\mathcal{L}_{abc}|_{a=0}$:*

$$\mathcal{L}_{0bc}\zeta_{bc}=0. \tag{37}$$

The spectrum of $\mathcal{L}_{0bc}$ starts with 0, which is a simple eigenvalue.

Proof. Since χ_λ is a static solution to (10), differentiating the equation $\Delta_r^{(0)}\chi_\lambda+r^{-1}\chi_\lambda\partial_r\chi_\lambda=0$ with respect to λ at $\lambda=1$, gives that

$$\zeta:=r\partial_r\chi=\frac{8r^2}{(1+r^2)^2} \tag{38}$$

is a zero mode of the linearization of (10) around χ:

$$\mathcal{L}_0\zeta=0, \tag{39}$$

where $\mathcal{L}_0:=-\Delta_r^{(0)}+r^{-1}\partial_r\cdot\chi$. (The vector ζ_λ spans the tangent space of the manifold $\mathcal{M}_0:=\{\chi_\lambda\mid\lambda>0\}$ at a point χ_λ.) We deform this result as (37). Consequently, by the Perron-Frobenius theorem, the spectrum of $\mathcal{L}_{0bc}$ starts with 0, which is a simple eigenvalue. □

The results above can be translated to the operator $L_{bc} := L_{abc}|_{a=0} := \gamma_{0bc}^{1/2}\mathcal{L}_{0bc}\gamma_{0bc}^{-1/2}$, which is explicitly given by

$$L_{bc} = -\Delta^{(4)} - \frac{8bc}{(c+y^2)^2} + \frac{4b(1-b)y^2}{(c+y^2)^2}. \tag{40}$$

This is a deformation in b and c of the operator

$$L_0 := -\Delta_y^{(4)} - \frac{8}{(1+y^2)^2}. \tag{41}$$

The operator L_{bc}, defined on the Hilbert space $L^2([0,\infty), y^3dy)$, is self-adjoint with spectrum $[0,\infty)$. The bottom of the spectrum, 0, is a simple eigenvalue,

$$L_{bc}\eta_{bc} = 0, \quad \text{with} \quad \eta_{bc} := 4\gamma_{0bc}^{1/2}\zeta_{bc} = \frac{1}{(c+y^2)^b}. \tag{42}$$

The tangent vectors $\zeta_{bc0}(y)$ and $\zeta_{bc1}(y)$ (to the manifold $\mathcal{M}$) are approximate eigenfunctions of the operator $\mathcal{L}_{abc}$. Indeed, first observe that the functions $\chi_{bc}(y)$ are approximate solution to (12). Indeed, let $\Phi(u)$ be the map defined by the right hand side of (12), $\Phi(u) := \Delta_y^{(0)}u + y^{-1}u\partial_y u - ay\partial_y u$, then

$$\Phi(\chi_{bc})(y) = -\frac{8bca}{c+y^2} + 8bc\frac{4(b-1)+ac}{(c+y^2)^2} - \frac{32bc^2(b-1)}{(c+y^2)^3}. \tag{43}$$

Now, differentiating $\Phi(\chi_{bc})$ with respect to c and b and using (43), we obtain

$$\begin{aligned}
\mathcal{L}_{abc}\zeta_{bc0}(y) &= -2a\zeta_{bc0}(y) + [4\frac{4(b-1)+ac}{c+y^2} - 24\frac{c(b-1)}{(c+y^2)^2}]\zeta_{bc0}(y),\\
\mathcal{L}_{abc}\zeta_{bc1}(y) &= [\frac{2ca}{c+y^2} - \frac{8c(2b-1)}{(c+y^2)^2}]\zeta_{bc1}(y).
\end{aligned} \tag{44}$$

Though on the first sight ζ_{bc0} and, especially, ζ_{bc1} do not seem to be approximate eigenfunctions of $\mathcal{L}_{abc}$, in fact they are. Indeed, assuming $b-1 = O(a\ln\frac{1}{a}), c = O(1)$, we obtain

$$\|(\mathcal{L}_{abc} + 2a\delta_{i0})\zeta_{bci}\| \lesssim \begin{cases} (b-1)+a & i=0,\\ 1 & i=1. \end{cases} \tag{45}$$

However, if one takes into account the normalizations

$$\|\zeta_{bc0}\| = \frac{1}{4\sqrt{2}}\ln^{\frac{1}{2}}\frac{1}{a} + O(\frac{1}{\ln^{\frac{1}{2}}\frac{1}{a}}), \qquad \|\zeta_{bc1}(y)\| = \frac{\sqrt{2}}{a} + O(\ln^2\frac{1}{a}), \tag{46}$$

then, for the normalized vectors we have

$$\|(\mathcal{L}_{abc} + 2a\delta_{i0})\frac{\zeta_{bci}}{\|\zeta_{bci}\|}\| \lesssim \begin{cases} \frac{(b-1)+a}{\sqrt{\ln\frac{1}{a}}} & i=0,\\ a & i=1. \end{cases} \tag{47}$$

The relation for $i=0$ implies (34). $\square$

4 Relation between Parameters a, b and c and Blowup Dynamics

In this section, we state the relations between the parameters a, b and c, which is obtained by evaluating the equations in (29) and is proven in Appendix D. Using these relations, we find the governing equation for $a(\tau)$.

Proposition 4.1. *Let $d := b - 1$ and assume*

$$\|\phi\| \ll (\ln \frac{1}{a})^{-1}\,, \tag{48}$$

and, for simplicity, $d \lesssim a\ln(a^{-1})$. Then

$$c_\tau + S_0(\phi, a, b, c)a_\tau = 4\big(\frac{ac}{2} - \frac{d}{\ln(\frac{1}{a})}\big) + O(\frac{a}{\ln(\frac{1}{a})}) + \mathcal{R}_0(\phi, a, b, c), \tag{49}$$

$$d_\tau + S_1(\phi, a, b, c)a_\tau = -\frac{2da}{\ln(\frac{1}{a})} + O(\frac{a^2}{\ln(\frac{1}{a})} + a^2 d\ln(\frac{1}{a})) + \mathcal{R}_1(\phi, a, b, c), \tag{50}$$

where $S_i(\phi, a, b, c)$ and $\mathcal{R}_i(\phi, a, b, c)$, $i = 0,\ 1$, satisfy the estimates

$$|S_i(\phi, a, b, c)| \lesssim \|\phi\|(a\ln(a^{-1}))^{i-1}, \tag{51}$$

$$|\mathcal{R}_i(\phi, a, b, c)| \lesssim \frac{a^{i+1}}{\ln^{1-i}(a^{-1})}\|\phi\| + \frac{a^i}{\ln(a^{-1})}\|\phi\|^2. \tag{52}$$

Remark 4.2. *We see from* (52) *that for the terms $\mathcal{R}_i(\phi, a, b, c)$ in* (49) - (50) *to be subleading we should have $\|\phi\| \ll (a\ln(a^{-1})^{-1})^{1/2}$.*

As was mentioned above, this proposition is proven in Appendix D. Now, we choose a relation between a, c and d, so that the leading order term on the right hand side of the equation for c_τ, (49), vanishes:

$$d = \frac{1}{2}a\ln(a^{-1})\,. \tag{53}$$

Proposition 4.3. *Assume $\|\phi\| \le \sqrt{\frac{a}{\ln(a^{-1})}}$ and* (53). *Then the function $a(\tau)$ satisfies the differential equation*

$$a_\tau = -\frac{2a^2}{\ln(a^{-1})}\big(1 + O(\frac{1}{\ln(a^{-1})})\big), \tag{54}$$

which gives

$$a(\tau) = \frac{\ln\tau}{2\tau}\big(1 + O(\frac{1}{\ln^2\tau})\big). \tag{55}$$

Proof. Plugging the relation (53) into (50) and remembering that $d = b - 1$, we obtain

$$\frac{1}{2}\big(\ln(a^{-1}) - 1 + 2S_2(\phi, a, b, c)\big)a_\tau = -\frac{2da}{\ln(\frac{1}{a})} + O(\frac{a^2}{\ln(\frac{1}{a})}) + \mathcal{R}_2(\phi, a, b, c), \tag{56}$$

We see that to solve this equation for a_τ, we need $|S_2| \ll \ln(1/a)$, which in view of (51) with $i = 2$ requires that $\|\phi\| \ll \ln(1/a)$. Due to the conditions of the proposition and estimate (52) with $i = 2$, the high order terms in equation (54) give a small correction upon integration and the leading part can be integrated exactly yielding (55). $\square$

Remark 4.4. *The above expression for a_τ passes a consistency test: $a_\tau < 0$ and $|a_\tau| \ll a^2$.*

Proposition 4.5. *For $|T-t| \ll 1$, the scaling parameter λ, with $a = -\lambda\dot{\lambda}$ satisfying (55), is asymptotic to*

$$\lambda = ke^{-\frac{1}{4}\ln^2\tau} \tag{57}$$

where τ is related to t by

$$k(T-t) = \frac{\tau}{\ln\tau}e^{-\frac{1}{2}\ln^2\tau}. \tag{58}$$

Proof. Using the definition $a = -\lambda\dot{\lambda}$ and the relation $\partial_t\lambda = \lambda^{-2}\partial_\tau\lambda$ we arrive at $a = -\lambda^{-1}\partial_\tau\lambda$. Combining this with (55) we obtain the equation $\lambda^{-1}\partial_\tau\lambda = -\dfrac{\ln\tau}{2\tau}\big(1 + O(\frac{1}{\ln^2\tau})\big)$. Solving this differential equation gives (57). Combining (57) with $\partial_t\tau = \lambda^{-2}$ gives a differential equation for $\tau(t)$ solving which in the leading order leads to (58). □

Solving (58) for $\ln^2\tau$ and substituting the result into (57) gives (9).

5 Lower Bound for the Operator $\mathcal{L}_{abc}$

In this section we investigate the linear operator $\mathcal{L}_{abc}$, defined in (21). The main result of this section is the following lower bound on the quadratic form $\langle\phi, \mathcal{L}_{abc}\phi\rangle$, $\phi \perp \zeta_{bci}$, where, recall, the vectors ζ_{bci} are defined in (28).

Theorem 5.1. *For $|a| \ll 1$, $|b-1| \ll 1$, $|a_\tau| \ll a^2$, $|b_\tau| \ll a(1-b)$ and for any $\phi \in H^1([0,\infty), \gamma_{abc}(y)y^3dy)$, $\phi\perp\zeta_{bci}$, $i = 0, 1$ we have, for some absolute constant $c > 0$,*

$$\langle\phi, \mathcal{L}_{abc}\phi\rangle \geq ca\,\|\phi\|^2_{H^1}\,. \tag{59}$$

Proof. Recall that the operator $\mathcal{L}_{abc}$ is unitarily equivalent to the operator L_{abc},

$$L_{abc} = \gamma_{abc}^{1/2}\mathcal{L}_{abc}\gamma_{abc}^{-1/2}\,, \tag{60}$$

acting on the space $L^2([0,\infty), y^3dy)$ with the inner product $(\phi,\eta) := \int\phi\eta y^3dy$. The latter operator has been explicitly computed in the proof of Proposition 3.1 to be

$$L_{abc} = L_* + W(y) - 2ab\,, \tag{61}$$

where

$$L_* := -\Delta_y^{(4)} - \frac{4bc(b+1)}{(c+y^2)^2} + \frac{4b(b-1)}{c+y^2} + \frac{1}{4}a^2y^2 \tag{62}$$

and

$$W(y) := \frac{2bac}{c+y^2} \geq 0\,.$$

Since the lower bound of L_{abc} is equal to the lower bound of $\mathcal{L}_{abc}$, we estimate the former lower bound. We observe that, like L_{abc}, the operator L_* is self-adjoint on $L^2([0,\infty),\, y^3dy)$ and its spectrum is purely

discrete, provided $a > 0$. The latter property follows from the fact that the potential in (62) goes to infinity as $y \to \infty$. Moreover, $L_* \geq 0$. Indeed, write $L_* = L_{0bc} + \frac{1}{4}a^2y^2$, where

$$L_{0bc} := -\Delta^{(4)} - \frac{4bc(b+1)}{(c+y^2)^2} + \frac{4b(b-1)}{c+y^2} .$$

Define $\eta_1(y) := \frac{1}{2\chi(y)}\zeta(y)$, where ζ is defined in (38), so that $L_{011}\eta_1 = 0$. We compute

$$\eta_1(y) := \frac{1}{2\chi(y)}\zeta(y) = \frac{1}{1+y^2}. \tag{63}$$

An extension of relation (63) leads to the equation

$$L_{0bc}\eta_{bc} = 0, \tag{64}$$

where η_{bc} is a deformation in b and c of η_1 given by

$$\eta_{bc} := \frac{1}{(c+y^2)^b}. \tag{65}$$

Since $\eta_{bc} > 0$ we conclude, as in Lemma 3.2, that $L_{0bc} \geq 0$ (with the zero being a resonance of L_{bc}). This, together with $L_* = L_{0bc} + \frac{1}{4}a^2y^2$, implies that $L_* \geq 0$. Next, we have

Lemma 5.2. *For $\phi \in H^2([0,\infty), y^3dy)$, $\phi \perp \zeta_{bci}$, $i = 0, 1$, and $a \ll 1$, we have*

$$(L_*\phi, \phi) \geq \big(4 - O(\frac{1}{\sqrt{\ln\frac{1}{a}}})\big)a\|\phi\|_*^2, \tag{66}$$

where $(\phi, \eta) := \int \phi\eta y^3 dy$ is the inner product in $L^2([0,\infty), y^3dy)$ and $\|\cdot\|_*$ is the corresponding norm.

Proof. To this end we will use the minimax principle for self-adjoint operators (see [74]), which states that the third eigenvalue, λ_3, of L_*

$$\lambda_3 = \inf_{\dim V=3} \max_{\phi\in V} \frac{(L_*\phi,\phi)}{\|\phi\|_*^2}, \tag{67}$$

where V is an arbitrary subspace of $H^1(\mathbb{R}^4)$ and the estimate from [30] of λ_3:

$$\lambda_3 = 4a + \frac{Ca}{\ln\frac{1}{a}}(1 + o(1)), \tag{68}$$

for some constant C. Now, let η be the minimizer to $\langle L_*\phi, \phi\rangle$ over

$$\{\phi \in H^2([0,\infty), y^3dy) \mid \phi\perp\zeta_{bci},\ i = 0, 1,\ \|\phi\|_* = 1\}.$$

Since L_* is selfadjoint, η can be chosen to be real. Since the spectrum of L_* is discrete, this minimizer exists. By the linear independence of ζ_{bci}, $i = 0, 1$ and orthogonality of η to ζ_{bci}, $i = 0, 1$, the three vectors η, ζ_{bci}, $i = 0, 1$ span a three dimensional space. The minimax principle then asserts that

$$\lambda_3 \leq \max_{\phi\in W} \frac{(L_*\phi,\phi)}{\|\phi\|_*^2}, \tag{69}$$

where $W := \text{span}\{\eta,\, \zeta_{bci},\ i = 0, 1\}$. Let $\phi_{na},\ n = 0, 1$, be an appropriate orthonormal basis in $\text{span}\{\zeta_{bci},\ i = 0, 1\}$:

$$\phi_{na} := \eta_{bc}\psi_{na}, \tag{70}$$

with

$$\psi_{0a} := \sqrt{\frac{2}{\ln\frac{1}{a}}}\big[1 + \mathrm{O}\big(\frac{1}{\ln\frac{1}{a}}\big)\big]e^{-\frac{a}{4}y^2},\ \psi_{1a} :=\ \big[1 + \mathrm{O}\big(\frac{1}{\ln\frac{1}{a}}\big)\big]\big(\frac{c_1}{\ln^{\frac{1}{2}}\frac{1}{a}} - c_2ay^2\big)e^{-\frac{a}{4}y^2}, \tag{71}$$

for some positive constants c_1 and c_2. We write $\phi = \gamma_1\eta + \gamma_2\phi_{0a} + \gamma_3\phi_{1a}$ in the inner product $(L_*\phi, \phi)$, where

$$|\gamma_1|^2 + |\gamma_2|^2 + |\gamma_3|^2 = 1, \tag{72}$$

and use self-adjointness of L_* to obtain that

$$\begin{aligned}(L_*\phi, \phi) =&|\gamma_1|^2(L_*\eta, \eta) + |\gamma_2|^2(L_*\phi_{0a}, \phi_{0a}) + |\gamma_3|^2(L_*\phi_{1a}, \phi_{1a})\\ &+ 2\,\mathrm{Re}\,(\gamma_1\gamma_2^*)(\eta, L_*\phi_{0a}) + 2\,\mathrm{Re}\,(\gamma_1\gamma_3^*)\,\langle\eta, L_*\phi_{1a}\rangle\\ &+ 2\,\mathrm{Re}\,(\gamma_2\gamma_3^*)(L_*\phi_{0a}, \phi_{1a}).\end{aligned} \tag{73}$$

We compute the various matrix elements on the right hand side of equation (73). Using that

$$\Delta_y^{(4)}\phi_{na} = \psi_{na}\Delta_y^{(4)}\eta_{bc} + \eta_{bc}\Delta_y^{(4)}\psi_{na} + 2\big(\partial_y\psi_{na}\big)\big(\partial_y\eta_{bc}\big),$$

we find

$$L_*\phi_{na} = \psi_{na}L_{0bc}\eta_{bc} + \eta_{bc}H_a\psi_{na} - 2\big(\partial_y\eta_{bc}\big)\big(\partial_y\psi_{na}\big).$$

Using the facts that $L_{0bc}\eta_{bc} = 0$, $H_a\psi_{0a} = 2a\psi_{0a}$ and $H_a\psi_{1a} = 4a\psi_{1a} + (8c_2a\sqrt{\frac{\ln\frac{1}{a}}{2}} - \sqrt{2}c_1a)\psi_{0a}$ and computing $2\big(\partial_y\eta_{bc}\big)\big(\partial_y\psi_{na}\big)$, we obtain that

$$\big(L_* - 2an\big)\phi_{na} = S_n,$$

with

$$S_0 := -\big(2a(b-1) + \frac{2abc}{c + y^2}\big)\phi_{0a}\,,$$

and

$$\begin{aligned}S_1 :=& -2a\big((b-1) - \frac{bc}{c+y^2}\big)\phi_{1a}\\ &- \sqrt{2}a\big(4c_2\sqrt{\ln\frac{1}{a}}(b-1) - \frac{4c_2bc}{c+y^2}\sqrt{\ln\frac{1}{a}} + 2c_1\big)\phi_{0a}\,.\end{aligned}$$

Using this and the fact that the functions ϕ_{ia} are normalized, we estimate

$$(L_*\phi_{0a}, \phi_{0a}) = -\frac{a}{2} + \mathrm{O}\big(\frac{a}{\ln\frac{1}{a}}\big)\,, \tag{74}$$

$$(L_*\phi_{0a}, \phi_{1a}) = -\frac{c_1 a}{\sqrt{2}\ln\frac{1}{a}} + \mathrm{O}\big(\frac{a}{\ln^{\frac{3}{2}}\frac{1}{a}}\big) \tag{75}$$

$$(L_*\phi_{1a}, \phi_{1a}) = \frac{a}{\ln^{\frac{1}{2}}\frac{1}{a}}(2 + \frac{c_1}{\sqrt{2}}) + \mathrm{O}\big(\frac{a}{\ln^2\frac{1}{a}}\big). \tag{76}$$

Let $P^\perp$ be the orthogonal projection onto the orthogonal complement of the two vectors ϕ_{0a} and ϕ_{1a}. We compute that

$$\|P^\perp L_*\phi_{ia}\|_* = \mathrm{O}\big(\frac{a}{\ln^{\frac{1}{2}}\frac{1}{a}}\big),\ i = 0, 1. \tag{77}$$

Using the estimates, (74) - (77), together with (69) and (73), we find

$$\lambda_3 \le \max_{\gamma_i}\big\{|\gamma_1|^2(L_*\eta, \eta) - \frac{a}{2}|\gamma_2|^2 + \mathrm{O}\big(\frac{a}{\ln^{\frac{1}{2}}\frac{1}{a}}\big)\big\}.$$

Now, since $L_* \ge 0$, we know that $(L_*\eta, \eta) \ge 0$. Then the above relation implies

$$\lambda_3 \le (L_*\eta, \eta) + \mathrm{O}\big(\frac{a}{\ln^{\frac{1}{2}}\frac{1}{a}}\big). \tag{78}$$

Using expression (68) for λ_3 in the last inequality, we obtain that

$$(L_*\eta, \eta) \ge 4\big(1 - O(\frac{1}{\sqrt{\ln\frac{1}{a}}})\big)a,$$

which gives the inequality (66). $\square$

Because of the decomposition (61) and since $W(y) \ge 0$ and $0 < b - 1 \lesssim \frac{1}{\sqrt{\ln\frac{1}{a}}}$, we arrive at

$$(\phi, L_{abc}\phi) \ge \frac{3}{2}a\|\phi\|_*^2\,, \tag{79}$$

or, by the unitary map $\phi \mapsto \gamma_{abc}^{1/2}\phi$,

$$\langle\phi, \mathcal{L}_{abc}\phi\rangle \ge \frac{3}{2}a\|\phi\|^2\,. \tag{80}$$

To pass from this bound to (59), we decompose $\langle\phi, \mathcal{L}_{abc}\phi\rangle = (1-\delta)\langle\phi, \mathcal{L}_{abc}\phi\rangle + \delta\langle\phi, \mathcal{L}_{abc}\phi\rangle$ and use (80) for the first term and $\mathcal{L}_{abc} \ge -\Delta^{(4)} - C$, for some $C > 0$, for the second one. Optimizing with respect to δ produces (59). $\square$

6 Analysis of Fluctuations

In this section, neglecting the nonlinearity $N(\phi)$, we find a bound $\|\phi\| \lesssim |1-b|$ on the fluctuation ϕ. Given that we expect, from (53), that $1-b \sim \frac{a}{2}\ln\frac{1}{a}$, this is sufficient to close the estimates. Neglecting the nonlinearity $N(\phi)$ in (20), we arrive at the linear equation

$$\partial_\tau \phi = -\mathcal{L}_{ab}\phi + \mathcal{F}_{ab}. \tag{81}$$

More precisely, we have the following proposition,

Proposition 6.1. *Assume a, b and ϕ solve* (81) *and* (29) *(with $\mathcal{N}=0$), which is equivalent to* (27)*, and are such that $1-b = \mathrm{O}\big(a\ln\frac{1}{a}\big)$ and $b_\tau = \mathrm{O}\big(a\big)$, and assume* (55) *holds. Then, for $\tau \gg 1$, ϕ satisfies the estimate*

$$\|\phi\| \lesssim \|\phi(0)\|\big(\frac{2a}{\ln\frac{1}{a}}\big)^{\ln\frac{1}{a}} + a\ln\frac{1}{a}.$$

Proof. We use a Lyapunov argument with Lyapunov functional $\phi \mapsto \|\phi\|^2$. The time derivative of this functional on solutions ϕ to (81) is

$$\partial_\tau\|\phi\|^2 = -2\,\langle\phi, \mathcal{L}_{abc}\phi\rangle + 2\,\langle\mathcal{F}_{abc}, \phi\rangle + \langle\phi, (\partial_\tau \ln\gamma_{abc})\phi\rangle \tag{82}$$

We estimate right hand side of this relation. Let $\chi(y)$, $\bar\chi(y) \geq 0$ be a smooth partition of unity, $\chi^2+\bar\chi^2 = 1$, s.t. $\chi(y)$ is a cutoff function that equals 1 on the set $\{ay^2 \leq \kappa\}$, for some convenient large constant $\kappa > 0$, and is supported on $\{ay^2 \leq 2\kappa\}$. We have

Proposition 6.2. *For any $\phi \in H^1([0,\infty), \gamma_{abc}(y)y^3\mathrm{d}y), \phi \perp \zeta_{bci}, i = 0,1$ we have, for some absolute constants $k_1, k_2 > 0$,*

$$2\langle\phi, \mathcal{L}_{abc}\phi\rangle \geq a\|\phi\|^2_{L^2} + k_1 a\|\phi\|^2_{H^1} + k_2\langle\bar\chi\phi, a^2y^2\bar\chi\phi\rangle\,. \tag{83}$$

Proof. Since ϕ is orthogonal to the vectors $\zeta_{bci}, i = 0,1$, and since a and b satisfy the conditions of Theorem 5.1 we have estimates (59) and (80), which imply

$$2\langle\phi, \mathcal{L}_{abc}\phi\rangle \geq \frac{3}{2}a\|\phi\|^2_{L^2} + ka\|\phi\|^2_{H^1}\,. \tag{84}$$

Next, we estimate $\langle\phi, \mathcal{L}_{abc}\phi\rangle$ in a different way. For the partition of unity defined after (82), we have the IMS formula (see e.g. [28])

$$\mathcal{L}_{abc} = \chi\mathcal{L}_{abc}\chi + \bar\chi\mathcal{L}_{abc}\bar\chi - |\nabla\chi|^2 - |\nabla\bar\chi|^2. \tag{85}$$

By (34) we have $\chi\mathcal{L}_{abc}\chi \gtrsim -a\chi^2$. Using the inner product $(\xi,\eta) := \int \xi\eta\mathrm{d}y$ and the notation $\bar\phi := \bar\chi\phi$ we obtain

$$\langle\bar\phi, \mathcal{L}_{abc}\bar\phi\rangle = (\bar\phi, \gamma_{abc}^{1/2}L_{abc}\gamma_{abc}^{1/2}\bar\phi)\,,$$

which, together with (36) gives, for κ large enough,

$$\langle\bar\phi, \mathcal{L}_{abc}\bar\phi\rangle \geq (\bar\phi, \gamma_{abc}^{1/2}[\frac{1}{4}a^2y^2 - 2ab - \frac{4bc(b+1)}{(c+y^2)^2}]\gamma_{abc}^{1/2}\bar\phi) \tag{86}$$

$$\geq \langle\bar\phi, (\frac{1}{8}a^2y^2 + \frac{1}{9}\kappa a)\bar\phi\rangle\,. \tag{87}$$

Next, using that $|\nabla\chi|$ and $|\nabla\bar{\chi}|$ are of the form $\sqrt{\frac{a}{\kappa}}\tilde{\chi}$, where $\tilde{\chi}$ is supported between $ay^2 = \kappa$ and $ay^2 = 2\kappa$, we compute $|\nabla\chi|^2 + |\nabla\bar{\chi}|^2 \simeq \frac{a}{\kappa}\tilde{\chi}$, which leads to

$$\langle \phi, (|\nabla\chi|^2 + |\nabla\bar{\chi}|^2)\phi\rangle \lesssim \frac{a}{\kappa}\|\tilde{\chi}\phi\|_{L^2}^2 . \tag{88}$$

Using the IMS formula (85) and the estimates above we find

$$\langle \phi, \mathcal{L}_{abc}\phi\rangle \geq -ca\|\chi\phi\|_{L^2}^2 + \frac{1}{9}\langle \bar{\chi}\phi, (a^2y^2 + \kappa a)\bar{\chi}\phi\rangle - C\frac{a}{\kappa}\|\phi\|_{L^2}^2 , \tag{89}$$

for positive constants c, C. Now, write $\langle \phi, \mathcal{L}_{abc}\phi\rangle = (1-\delta)\langle \phi, \mathcal{L}_{abc}\phi\rangle + \delta\langle \phi, \mathcal{L}_{abc}\phi\rangle$, and use (84) for the first term on the right hand side and (89) for the second one, and choose δ sufficiently small to arrive at (83). □

We substitute expression (22) for $\mathcal{F}_{abc}$ and observe that the orthogonality of ϕ to ζ_{bci}, $i = 0, 1$, implies

$$\langle 1, \phi\rangle = c\left\langle \frac{1}{c+y^2}, \phi\right\rangle = c^2\left\langle \frac{1}{(c+y^2)^2}, \phi\right\rangle .$$

to obtain

$$\langle \mathcal{F}_{abc}, \phi\rangle = 32bc(b-1)[\left\langle \frac{1}{(c+y^2)^2}, \phi\right\rangle - \left\langle \frac{1}{(c+y^2)^3}, \phi\right\rangle] .$$

Using Hölder's inequality in the above equality implies

$$\langle \mathcal{F}_{abc}, \phi\rangle = \mathrm{O}\big((b-1)\ \|\langle y\rangle^{4-\varepsilon}\phi\|\big). \tag{90}$$

Next, we estimate $\langle \phi, (\partial_\tau \ln\gamma_{abc})\phi\rangle$. Using that $\partial_\tau \ln\gamma_{abc} = -a_\tau y^2/2 + 2b_\tau \ln(c+y^2) + 2bc_\tau/(c+y^2)$, we obtain

$$\langle \phi, (\partial_\tau \ln\gamma_{abc})\phi\rangle = -\frac{1}{2}a_\tau\|y\phi\|_{L^2}^2 + 2b_\tau\|(\ln(c+y^2))^{1/2}\phi\|_{L^2}^2 + 2bc_\tau\|\frac{1}{\sqrt{c+y}}\phi\|_{L^2}^2 . \tag{91}$$

By (50) and (54), we have $a_\tau < 0$, $b_\tau < 0$, and assuming $c < 1$, we have by (49) and (53), that $c_\tau < 0$. Hence

$$\langle \phi, (\partial_\tau \ln\gamma_{abc})\phi\rangle \leq -\frac{1}{2}a_\tau\|y\phi\|_{L^2}^2 . \tag{92}$$

Now, $y^2 \leq \kappa/a$ on suppχ, which implies $\langle \chi\phi, (\partial_\tau \ln\gamma_{abc})\chi\phi\rangle \leq -\frac{a_\tau\kappa}{2a}\|\chi\phi\|_{L^2}^2$. This, together with the relation $\langle \chi\phi, (\partial_\tau \ln\gamma_{abc})\chi\phi\rangle = \langle \chi\phi, (\partial_\tau \ln\gamma_{abc})\chi\phi\rangle + \langle \bar{\chi}\phi, (\partial_\tau \ln\gamma_{abc})\bar{\chi}\phi\rangle$, gives

$$\langle \phi, (\partial_\tau \ln\gamma_{abc})\phi\rangle \leq -\frac{a_\tau\kappa}{2a}\|\chi\phi\|_{L^2}^2 - \frac{1}{2}a_\tau\|y\bar{\chi}\phi\|_{L^2}^2 . \tag{93}$$

Using the last estimate, together with (82), (83), (90), (93), (53) and (55), we obtain, for some absolute constants $k_1, k_2, C > 0$,

$$\partial_\tau\|\phi\|_{L^2}^2 \leq -a\|\phi\|_{L^2}^2 - k_1 a\|\phi\|_{H^1}^2 - k_2\langle \bar{\chi}\phi, (a^2y^2 + \kappa a)\bar{\chi}\phi\rangle + C(a\ln\frac{1}{a})\|\langle y\rangle^{-\frac{7}{4}+\varepsilon}\phi\|_{L^2}^2 .$$

Using $\partial_\tau \|\phi\|^2 = 2\|\phi\|\partial_\tau\|\phi\|$, dropping the second and third terms (these terms can be used to control the nonlinearity) and dividing the resulting inequality by $\|\phi\|$, we obtain

$$\partial_\tau \|\phi\| \le -\frac{a}{2}\|\phi\| + Ca\ln\frac{1}{a}. \tag{94}$$

Now, integrating the last inequality gives that

$$\|\phi\| \lesssim e^{-\int_0^\tau a(s)\,ds}\|\phi(0)\| + \int_0^\tau e^{-\int_\sigma^\tau a(s)\,ds}\big(a\ln\frac{1}{a}\big)(\sigma)\,d\sigma. \tag{95}$$

We have computed that in the sense of asymptotic equivalence, $a(\tau) \sim \frac{\ln\tau}{2\tau}$, as $\tau\to\infty$ (see equation (55)). Consequently, as $\sigma\to\infty$, we compute that

$$\int_\sigma^\tau a(s)\,ds \sim \frac{1}{4}\ln(\tau\sigma)\ln\big(\frac{\tau}{\sigma}\big) = \ln\big(\frac{\tau}{\sigma}\big)^{\frac{1}{4}\ln(\tau\sigma)}.$$

Using

$$\mathrm{e}^{-\int_0^\tau a(s)} = \mathrm{e}^{-\int_0^{\sqrt{\tau}} a(s)}\mathrm{e}^{-\int_{\sqrt{\tau}}^\tau a(s)},$$

noting that the first term on the right hand side is uniformly bounded, and the second term is $\sim \tau^{-\frac{3}{16}\ln\tau}$, we obtain $\mathrm{e}^{-\int_0^\tau a(s)} = O(\tau^{-\frac{3}{16}\ln\tau})$. Using now

$$\tau = \frac{\ln\frac{1}{a}}{2a}\big(1 - O(\frac{1}{\ln\frac{1}{a}})\big)$$

we obtain that the term involving the initial condition in (95) is bounded as

$$e^{-\int_0^t a(s)\,ds}\|\phi(0)\| \lesssim \|\phi(0)\|\big(\frac{2a}{\ln\frac{1}{a}}\big)^{\ln\frac{1}{a}}. \tag{96}$$

To bound the integral term in (95) we begin by splitting the domain of integration into $[0,\alpha\tau]$ and $[\alpha\tau,\tau]$ for some $0<\alpha<1$ to be chosen later:

$$\int_0^{\alpha\tau} e^{-\int_\sigma^\tau a(s)\,ds}\big(a\ln\frac{1}{a}\big)(\sigma)\,d\sigma + \int_{\alpha\tau}^\tau e^{-\int_\sigma^\tau a(s)\,ds}\big(a\ln\frac{1}{a}\big)(\sigma)\,d\sigma.$$

Since $(a\ln\frac{1}{a})(\sigma)$ is decreasing and $\sigma\mapsto e^{-\int_\sigma^\tau a(s)\,ds}$ is increasing and both are positive, we can bound these terms from above by

$$e^{-\int_{\alpha\tau}^\tau a(s)\,ds}\int_0^{\alpha\tau}\big(a\ln\frac{1}{a}\big)(\sigma)\,\mathrm{d}\sigma + \big(a\ln\frac{1}{a}\big)(\alpha\tau)\int_{\alpha\tau}^\tau e^{-\int_\sigma^\tau a(s)\,ds}\mathrm{d}\sigma\,.$$

Since $e^{-\int_{\alpha\tau}^\tau a(s)\,ds} \sim C_\alpha\tau^{\frac{1}{2}\ln\alpha}$, the first term is bounded from above by $C_\alpha\big(a\ln\frac{1}{a}\big)(0)\tau^{1+\frac{1}{2}\ln\alpha}$. Taking α such that $\ln(\alpha)/2 < -2$ the first term is $\lesssim \tau^{-1}$, and the second is bounded by $\big(a\ln\frac{1}{a}\big)(\alpha\tau) \sim C\ln^2\tau/\tau$. So we find

$$\int_0^\tau e^{-\int_\sigma^\tau a(s)\,ds}(a\ln\frac{1}{a})(\sigma)\,d\sigma \lesssim \big(a\ln\frac{1}{a}\big)(\tau). \tag{97}$$

Using bounds (96) and (97) in (95) completes the proof. $\square$

A Complete Set of Static Solutions for the Radial rKS

The static solutions of equation (10) satisfy the second order differential equation

$$\partial_r^2 \chi + \frac{1}{r}(\chi - 1)\partial_r \chi = 0 \tag{98}$$

and hence form a two dimensional manifold. We prove

Proposition A.1. *Equation* (98) *has the one-parameter family of static solutions*

$$\chi^{(\mu)}(r) := \frac{r^{\mu-2}\mu + 4 - \mu}{1 + r^{\mu-2}},\ \mu \in [2, \infty),$$

(and therefore the two-parameter family $\chi_\lambda^{(\mu)}(r) := \chi^{(\mu)}(r/\lambda)$ as well). The mass at infinity of $\chi_\lambda^{(\mu)}$ is μ.

Remark A.2. *If $\mu < 4$, then the mass at the origin is non-zero, i.e. blowup has already occurred. If $\mu > 4$, then the mass at the origin is negative and hence the static solution is not physical.*

Proof. We use the transformation

$$\psi(\chi) = r\frac{\partial_r \chi}{\chi}$$

in (98) under the assumption that the right hand side is indeed a function of χ alone. Using this transformation, equation (98) becomes

$$\chi \partial_\chi \psi + \psi = 2 - \chi.$$

Integrating this equation gives that

$$\psi = 2 - \frac{1}{2}\chi + \frac{\mu}{2}\frac{1}{\chi}$$

and hence, upon substituting this into the definition of ψ and integrating over r, we obtain the general solution

$$\chi = \frac{\left(\frac{r}{\lambda}\right)^{\sqrt{4+\nu}} r_+ + r_-}{1 + \left(\frac{r}{\lambda}\right)^{\sqrt{4+\nu}}},$$

where $r_\pm = 2 \pm \sqrt{4 + \nu}$ are the roots of $\chi^2 - 4\chi - \nu = 0$. The total mass at infinity of these solutions is r_+ and hence it is natural to define a new parameter $\mu = r_+ \in [2, \infty)$. The static solution in terms of the parameters λ and ν are

$$\chi = \frac{\left(\frac{r}{\lambda}\right)^{\mu-2}\mu + 4 - \mu}{1 + \left(\frac{r}{\lambda}\right)^{\mu-2}}$$

The constant λ is positive since it is the exponential of the constant obtained in the last integration. □

The tangent space of the manifold $M_{\lambda,\mu}$ is spanned by the functions

$$\zeta^0_{\lambda,\mu} := \partial_\lambda \chi_{\lambda,\mu} = -\frac{2(\mu-2)^2}{\lambda}\frac{y^{\mu-2}}{(1+y^{\mu-2})^2}$$

and

$$\zeta^1_{\lambda,\mu} := \partial_\mu \chi_{\lambda,\mu} = \frac{y^{2(\mu-2)} - 1 - 2y^{\mu-2}\ln y}{(1+y^{\mu-2})^2},$$

where $y = \frac{r}{\lambda}$.

We again restrict to the situation of $\kappa = 1$ and $n = 4$. After the gauge transform the zero modes $\zeta^0_{\lambda,\mu}$ and $\zeta^1_{\lambda,\mu}$ transform to

$$\eta^\lambda_{\lambda,\mu} := \frac{1}{\lambda^2}\frac{1+y^2}{y^2}\zeta^0_{\lambda,\mu} \text{ and } \eta^\mu_{\lambda,\mu} := \frac{1}{\lambda^2}\frac{1+y^2}{y^2}\zeta^1_{\lambda,\mu},$$

neither of which are in $L^2(r^3\,dr)$ and hence are generalized eigenfunctions of $\mathcal{L}$ (without the $\dot\lambda$ term. By the ODE theory the above functions are the only linearly independent solutions to the equation $\mathcal{L}_0\phi = 0$. The Perron-Frobenius theory shows that 0 is the lowest point of the spectrum of $\mathcal{L}$.

B Proof of Proposition 2.1

Both existence and uniqueness follow from a standard implicit function theorem argument. Fix $0 < \delta \ll 1$ and let $Z := e^{\frac{\delta}{3}y^2}L^\infty([0,\infty))$. Recall that $b = 1 + 1/2a\log(1/a)$, and define the vector-valued function

$$G(f,a,c) := \big(\langle f - \chi_{bc}, \zeta_{bci}\rangle\,,\ i = 0,1\big) \tag{99}$$

$$= \Big(\frac{1}{16}\int_0^\infty \big(f(y) - \chi_{bc}(y)\big)(c+y^2)^{2b-2+i} y e^{-\frac{a}{2}y^2}\,\mathrm{d}y,\ i = 0,1\Big)\,. \tag{100}$$

This function maps $Z \times \mathbb{R}_+ \times \mathbb{R}_+$ into $\mathbb{R}^2$. It is a C^1 function and $G(\chi_{bc}, a, c) = 0$. Moreover, the derivative of G with respect to (a,c) at $f = \chi_{bc}$ is

$$A := \begin{pmatrix} \Gamma_{0a} & \Gamma_{0c} \\ \Gamma_{1a} & \Gamma_{1c} \end{pmatrix}, \tag{101}$$

where

$$\Gamma_{ia} := \frac{\partial_a b}{4}\int_0^\infty y^3(c+y^2)^{2b-3+i}\mathrm{e}^{-\frac{a}{2}y^2}\,\mathrm{d}y \tag{102}$$

$$\Gamma_{ic} := -\frac{b}{4}\int_0^\infty y^3(c+y^2)^{2b-4+i}\mathrm{e}^{-\frac{a}{2}y^2}\,\mathrm{d}y\,. \tag{103}$$

Compute that the determinant of A satisfies

$$|\det A| = \frac{1}{64a^2}(\ln^2\frac{1}{a} + O(1)),$$

as $a \to 0$, and so $|\det A| \geq C > 0$, for some constant C, for $(a, c) \in (0, \delta) \times (1, 2)$ for δ small enough. Thus by the implicit function theorem, for any $a_* \in (0, \delta)$ and $c_* \in (1, 2)$ there exist open sets $U_{a_* c_*} \subset Z$ and $V_{a_* c_*} \subset (0, \delta) \times (1, 2)$ containing $\chi_{b_* c_*}$ and (a_*, c_*), respectively, and a unique function $g_{a_* c_*} : U_{a_* c_*} \to V_{a_* c_*}$. To determine the size of the neighbourhoods $U_{a_* c_*}$ we look more closely into a proof of the implicit function theorem. Write $\mu = (a, c)$ and expand

$$G(f, \mu) = G(f, \mu_*) + \partial_\mu G(f, \mu_*)(\mu - \mu_*) + R_f(\mu)\,, \tag{104}$$

where $R_f(\mu) = O(|\mu - \mu_*|^2)$ uniformly in $f \in B_C(\chi_{b_* c_*})$ and $(a_*, c_*) \in (\delta/2, \delta) \times (1, 2)$, for any fixed constant C. By continuity and the above computations, there is $\varepsilon > 0$ such that $\det \partial_\mu G(f, \mu_*)$ is bounded away from zero uniformly for $f \in B_\varepsilon(\chi_{b_* c_*})$ and $(a_*, c_*) \in (\delta/2, \delta) \times (1, 2)$. From (104) we find a fixed-point equation for $\mu - \mu_*$,

$$\mu - \mu_* = \Phi_f(\mu - \mu_*)\,,$$

where

$$\Phi_f(\mu) = -(\partial_\mu G(f, \mu_*))^{-1}(G(f, \mu_*) + R_f(\mu))\,.$$

The above observations imply that there is an $\varepsilon_1 > 0$ such that Φ_f is a contraction on $B_{\varepsilon_1}(\mu_*)$ for any $f \in B_\varepsilon(\chi_{b_* c_*}) =: U_{a_* c_*}$. Taking the union of U_{ac} over $a \in (\delta, 1)$ and $c \in (1/2, 1)$ gives the open set $\mathcal{U}_\varepsilon$. Patching together the functions g_{ac} gives g. $\square$

C Gradient Formulation

The Keller-Segel models (1) and (4) are gradient systems. We begin by formulating a normalized version of (1),

$$\begin{aligned} \partial_t \rho &= \Delta\rho - \nabla \cdot (f(\rho)\nabla c) \\ \varepsilon \partial_t c &= \Delta c + \rho - \gamma c, \end{aligned} \tag{105}$$

as a gradient system. This system is obtained from (1) by setting unimportant constants to 1.

Define the energy (or Lyapunov) functional

$$\mathcal{E}_f(\rho, c) := \int_\Omega \frac{1}{2}|\nabla c|^2 - \rho c + \frac{\gamma}{2}c^2 + G(\rho)\, dx, \tag{106}$$

where $G(\rho) := \int^\rho g(s)\, ds$ and $g(\rho) := \int^\rho \frac{1}{f(s)}\, ds$. The L^2-gradient of $\mathcal{E}_f(\rho, c)$ is

$$\operatorname{grad}_{L^2} \mathcal{E}_f(\rho, c) = \begin{pmatrix} -c + g(\rho) \\ -\Delta c - \rho + \gamma c \end{pmatrix},$$

and hence, if we define $U = (\rho, c)$, then (105) can be written in the form $\partial_t U = I\mathcal{E}_f'(U)$, where

$$I = \begin{pmatrix} \nabla \cdot f(\rho)\nabla & 0 \\ 0 & -\frac{1}{\varepsilon} \end{pmatrix}.$$

The operator I is non-positive and may be degenerate, however, assuming it is invertible, the operator I defines the metric $\langle v, w\rangle_I := -\left\langle v, I^{-1}w\right\rangle_{L^2\oplus L^2}$. In this metric, $\operatorname{grad}\mathcal{E}(U) = -I\mathcal{E}'(U)$ and hence

$$\partial_t U = -\operatorname{grad}\mathcal{E}_f(U).$$

This shows that (105) has the structure of a gradient system. A consequence of this is that the energy decreases on solutions of the KS system. Indeed, if $f > 0$, then

$$\partial_t \mathcal{E}_f(\rho, c) = -\left\|f(\rho)^{\frac{1}{2}}\nabla\big(c - g(\rho)\big)\right\|^2 - \frac{1}{\varepsilon}\left\|\Delta c + \rho - c\right\|^2.$$

The gradient formulation for (4) is similar to the one for (105). Instead of (106), one uses the energy (7). The latter is obtained from (106) by dropping the quadratic term $\frac{1}{2}c^2$, replacing c with $-\Delta^{-1}\rho$ in the remaining terms and using that $f(\rho) = \rho$. The formal Gâteaux derivative of $\mathcal{E}$ is $\partial_\rho\mathcal{E}(\rho)\phi = \int(\Delta^{-1}\rho + \ln\rho)\phi$, and therefore the gradient in the metric $\langle v, w\rangle_J := -\left\langle v, J^{-1}w\right\rangle_{L^2}$, where $J := \nabla\cdot\rho\nabla < 0$, is

$$\operatorname{grad}\mathcal{E}(\rho) = -\nabla\cdot\rho\nabla(\Delta^{-1}\rho + \ln\rho) = -\nabla\cdot\rho\nabla\Delta^{-1}\rho - \Delta\rho,$$

which is the negative of the r.h.s. of the first equation in (4) with $c = -\Delta^{-1}\rho$. Hence the equation (4) can be written as $\partial_t\rho = -\operatorname{grad}\mathcal{E}(\rho)$ in the space with metric $\langle v, w\rangle_J := -\left\langle v, J^{-1}w\right\rangle_{L^2}$. Again, the energy $\mathcal{E}$ decreases on solutions of (4):

$$\partial_t\mathcal{E} = \langle\mathcal{E}', I\mathcal{E}'\rangle = -\left\|\rho^{\frac{1}{2}}\nabla\mathcal{E}'\right\|^2. \tag{107}$$

This can be thought of as an entropy monotonicity formula.

The stationary solutions of (4) are critical points of the energy functional $\mathcal{E}$, given in (7), under the constraint that $\int\rho = const.$. Thus, they satisfy $\mathcal{E}'(\rho) = C$, where C is a constant. Explicitly $\mathcal{E}'(\rho) = C$ reads

$$\log(\rho) + \frac{1}{\Delta}\rho = C \;\Leftrightarrow\; \Delta\log(\rho) + \rho = 0 \;\Leftrightarrow\; \Delta u + e^u = 0, \tag{108}$$

where $u = \log(\rho)$. Solutions to (108) can be written in the form of 'Gibbs states' $\rho = M\frac{e^c}{\int e^c}$ (see [33]), with the concentration c considered as a negative potential (remember that $\Delta c = -\rho$). In two dimensions, this equation has the solution $R = \frac{8}{(1+|x|^2)^2}$. This solution is a minimizer of $\mathcal{E}$ under the constraint that $\int\rho = 8\pi$.

D Proof of Proposition 4.1

In this appendix, we prove Proposition 4.1, relating the parameters a, b and c, by evaluating the equations in (29).

Proof of Proposition 4.1. Let $R_i(\phi) := \langle\mathcal{L}_{abc}\phi, \zeta_{bci}\rangle - \langle\mathcal{N}, \zeta_{bci}\rangle$. Here and in what follows $i = 0, 1$. The equations (29) can be rewritten as

$$\langle\mathcal{F}_{abc}, \zeta_{bci}\rangle + \langle\phi, \partial_\tau\zeta_{bci} + (\partial_\tau\ln\gamma_{ab})\zeta_{bci}\rangle = R_i(\phi). \tag{109}$$

We begin with evaluating $\langle \mathcal{F}_{abc}, \zeta_{bci}\rangle$ to *leading order*. To this end, we begin with the elementary computation

$$\langle 1, \zeta_{bci}\rangle = 2^{i-4}a^{-i-1} + O(\frac{1}{a^i}\ln^2\frac{1}{a}), \tag{110}$$

$$\left\langle \frac{1}{c+y^2}, \zeta_{bci}\right\rangle = 2^{i-5}a^{-i}\ln^{1-i}\frac{1}{a} + O(\ln^{2i}\frac{1}{a}), \tag{111}$$

$$\left\langle \frac{1}{(c+y^2)^2}, \zeta_{bci}\right\rangle = 2^{-5}c^{i-1}\ln^{i}\frac{1}{a} + O(a^{1-i}\ln^{1-i}\frac{1}{a}), \tag{112}$$

$$\left\langle \frac{1}{(c+y^2)^3}, \zeta_{bci}\right\rangle = 2^{i-6}c^{i-2} + O(a\ln\frac{1}{a}). \tag{113}$$

These estimates are proven at the end of this appendix. Using these estimates in (22), we arrive at

$$\begin{aligned}\langle \mathcal{F}_{abc}, \zeta_{bci}\rangle &= -b_\tau 2^{i-2}a^{-i-1}\big(1 - c2^{-1}a\ln^{1-i}\frac{1}{a}\big) + c_\tau\big(b2^{i-3}a^{-i}\ln^{1-i}\frac{1}{a} - bc^i 2^{-3}\ln^i\frac{1}{a}\big)\\ &- 2^{i-2}bca^{1-i}\ln^{1-i}\frac{1}{a} + (bc^i d + abc^{i+1}2^{-2})\ln^i\frac{1}{a} - bc^i d 2^{i-1}\\ &+ O(b_\tau a^{-i}\ln^2\frac{1}{a}) + O((a_\tau + a)\ln^{2i}\frac{1}{a}) + O((d + a + c_\tau)a^{1-i}\ln^{1-i}\frac{1}{a}) + O(da\ln\frac{1}{a})\,.\end{aligned} \tag{114}$$

Next, we compute the term $\langle \phi, \partial_\tau\zeta_{bci} + (\partial_\tau \ln\gamma_{abc})\zeta_{bci}\rangle$. Differentiating ζ_{bci} and $\partial_\tau\ln\gamma_{ab}$ with respect to τ, we obtain

$$\partial_\tau\zeta_{bci} + (\partial_\tau\ln\gamma_{abc})\zeta_{bci} = \big[2b_\tau\ln(c+y^2) + (2b - 2^{1-i})\frac{c_\tau}{c+y^2} - \frac{a_\tau}{2}y^2\big]\zeta_{bci}. \tag{115}$$

Using in the case $i=1$ that ϕ is orthogonal to ζ_{bc0}, we find

$$\langle \phi, \partial_\tau\zeta_{bci} + (\partial_\tau\ln\gamma_{ab})\zeta_{bci}\rangle = b_\tau S_{i1}(\phi) + c_\tau S_{i2}(\phi) - a_\tau S_{i3}(\phi), \tag{116}$$

where

$$\begin{aligned}S_{i1}(\phi) &:= 2\left\langle \phi, \ln(c+y^2)\zeta_{bci}\right\rangle\\ S_{i2}(\phi) &:= 2\delta_{i0}(b-1)\left\langle \phi, \frac{1}{c+y^2}\zeta_{bci}\right\rangle\\ S_{i3}(\phi) &:= \frac{1}{2}\left\langle \phi, y^2\zeta_{bci}\right\rangle.\end{aligned}$$

Collecting (114) and (116), we have, for $i=0$,

$$\begin{aligned}&(S_{03}(\phi) + O(1))a_\tau + [\frac{1}{4a} - \frac{c}{8}\ln(a^{-1}) - S_{01}(\phi) + O(\ln^2\frac{1}{a})]b_\tau\\ &- [\frac{b}{8}(\ln(a^{-1}) - 1) + S_{02}(\phi) + O(a\ln\frac{1}{a})]c_\tau = -\frac{b}{2}[\frac{1}{2}ac\ln(a^{-1}) - d - ac] - R_0(\phi)\,.\end{aligned} \tag{117}$$

and, for $i=1$,

$$\begin{aligned}(S_{13}(\phi)a + O(\ln^2\frac{1}{a}))a_\tau + [\frac{1}{2a} - \frac{c}{4} - aS_{11}(\phi) + O(a^{-1}\ln^2\frac{1}{a})]b_\tau - [\frac{b}{4} - \frac{1}{8}abc\ln(a^{-1}) + aS_{12}(\phi) + O(1)]c_\tau\\ = -abc[\frac{1}{2} - d\ln(a^{-1}) - \frac{ac}{4}\ln(a^{-1}) + d] - aR_1(\phi)\,.\end{aligned} \tag{118}$$

We manipulate equations (117) and (118) and solve them for b_τ and c_τ to obtain

$$-f_0a_\tau + gc_\tau = \frac{1}{8}bc\ln(a^{-1}) - \frac{bd}{4a} + v_0 + r_0, \quad -f_1a_\tau + gb_\tau = -\frac{b^2d}{8} + v_1 + r_1, \tag{119}$$

where

$$\begin{aligned}
f_0 &:= (\frac{1}{2a} - \frac{c}{4} - aS_{11})S_{03} + (\frac{1}{4} + \frac{c}{8}a\ln(a^{-1}) - aS_{01})S_{13},\\
g &:= \frac{b}{16}\frac{\ln(a^{-1})}{a} - \frac{1}{4}\big(\frac{b}{2a} - \frac{bc}{8} - \frac{1}{16}bc^2a\ln(a^{-1})^2 - \frac{1}{8}bc\ln(a^{-1}) - (\frac{2}{a} - c)S_{02} - bS_{01} + S_{12}\big)\\
&\quad - \frac{a}{8}\ln(a^{-1})\big(bcS_{01} + bS_{11} - cS_{12}\big) + a\big(\frac{1}{8}bS_{11} - S_{02}S_{11} + S_{01}S_{12}\big),\\
v_0 &:= \frac{1}{8}bc[2d\ln(a^{-1}) + 3 + d + ca\ln(a^{-1})(d\ln(a^{-1}) - \frac{1}{2} - d) - ca - \frac{1}{4}a^2c^2\ln(a^{-1})^2],
\end{aligned}$$

$$\begin{aligned}
r_0 &:= (\frac{1}{2a} - \frac{1}{4}c)R_0 - \frac{1}{4}R_1 + ca\ln(a^{-1})(bdS_{01} - \frac{1}{8}R_1)\\
&\quad + a\big(\frac{1}{2}bcS_{01} + \frac{1}{2}bdS_{11} + R_0S_{11} + S_{01}R_1\big) + \frac{1}{2}a^2bc(S_{11} - \frac{1}{2}\ln(a^{-1})(cS_{01} + S_{11})),\\
f_1 &:= a(\frac{b}{8}\ln(a^{-1}) - \frac{b}{8} + S_{02})S_{13} - (\frac{b}{4} - \frac{bc}{8}a\ln(a^{-1}) + aS_{12})S_{03},
\end{aligned}$$

$$v_1 := -\frac{1}{8}b^2c[da\ln^2(a^{-1}) - \frac{3}{2}da\ln(a^{-1}) - a(\frac{1}{2} - d) + \frac{1}{4}ca^2\ln(a^{-1})],$$

$$\begin{aligned}
r_1 &:= -\frac{b}{4}R_0 + b(\frac{1}{8}cR_0 + -dcS_{02} + \frac{1}{8}R_1)a\ln(a^{-1}),\\
&\quad + a\big(\frac{bcS_{02}}{2} + bcdS_{02} + \frac{bdS_{12}}{2} - R_0S_{12} - \frac{1}{8}bR_1 + R_1S_{02}\big),\\
&\quad + \frac{1}{2}bca^2\big(S_{12} + \frac{1}{2}cS_{02}\ln(a^{-1}) + \frac{1}{2}S_{12}\ln(a^{-1})\big).
\end{aligned}$$

Next, we derive estimates on $S_{i1}(\phi)$, $S_{i2}(\phi)$, $R_0(\phi)$ and $R_1(\phi)$. Using the Cauchy-Schwarz inequality and simple modifications of the estimates (46), we arrive at the estimates

$$|S_{i1}(\phi)| \lesssim \|\phi\|a^{-i}\ln(a^{-1})^{(3-i)/2},\ |S_{i2}(\phi)| \lesssim \|\phi\|\ln(a^{-1})^{\frac{i}{2}},\ |S_{i3}(\phi)| \lesssim \|\phi\|a^{-(i+1)}. \tag{120}$$

As was shown above, the operator $\mathcal{L}_{abc}$ is self-adjoint in the inner product (25) and hence $\langle \mathcal{L}_{abc}\phi, \zeta_{bci}\rangle = \langle \phi, \mathcal{L}_{abc}\zeta_{bci}\rangle$. Using (45) and the fact that ζ_{bc0} is orthogonal to ϕ, we obtain the estimate

$$|\langle \mathcal{L}_{abc}\phi, \zeta_{bci}\rangle| \lesssim \|\phi\|(d+a)^{1-i}. \tag{121}$$

Lastly, we estimate $\langle \mathcal{N}, \zeta_{bci}\rangle$ which can be written, using integration by parts, in the form

$$\langle \mathcal{N}, \zeta_{bci}\rangle = -\frac{1}{2}\int_0^\infty \phi^2\partial_y(\gamma_{abc}y^2\zeta_{bci})\,dy,$$

where, recall, γ_{abc} is the gauge function (see (24)). Here we used that $y\phi \to 0$ as $y \to \infty$ so that the boundary terms vanish. Using that

$$\partial_y(\gamma_{abc}y^2\zeta_{bci}) = \partial_y(\frac{(c+y^2)^{2b}}{16y^2}e^{-\frac{a}{2}y^2}\zeta_{bci}) = [(\frac{4b}{c+y^2} - \frac{2}{y^2} - a)\zeta_{bci} + y^{-1}\partial_y\zeta_{bci}]\gamma_{abc}y^3,$$

we find

$$|\langle \mathcal{N}, \zeta_{bci}\rangle| \lesssim \big\|(c+y^2)^{-\frac{2-i}{2}}\phi\big\|^2 + a\big\|(c+y^2)^{-\frac{1-i}{2}}\phi\big\|^2. \tag{122}$$

Estimates (121) and (122) give

$$|R_i(\phi)| \lesssim (d+a)^{1-i}\|\phi\| + \big\|(c+y^2)^{-\frac{2-i}{2}}\phi\big\|^2. \tag{123}$$

Since we assumed $d \lesssim a\ln(a^{-1})$, the above estimates imply the following inequalities for f_i and r_i

$$|f_i| \lesssim a^{i-2}(\ln\frac{1}{a})^i\|\phi\|, \tag{124}$$

$$\begin{aligned}|r_i| &\lesssim a^{i-1}|R_0(\phi)| + (a\ln(a^{-1}))^i|R_1(\phi)| \lesssim a^{i-1}[(d+a)\|\phi\| + \big\|(c+y^2)^{-1}\phi\big\|^2] \\ &\quad + (a\ln(a^{-1}))^i[\|\phi\| + \big\|(c+y^2)^{-\frac{1}{2}}\phi\big\|^2] \\ &\lesssim (a\ln(a^{-1}))^i\|\phi\| + a^{i-1}\|\phi\|^2,\end{aligned} \tag{125}$$

The estimates (120) show that $g = \frac{\ln(a^{-1})}{a}(1+o(1))$, provided

$$a^{-1}|S_{02}|,\ |S_{01}|,\ |S_{02}|,\ |S_{12}|,\ a\ln(a^{-1})|S_{11}|,\ a|S_{02}||S_{11}|,\ a|S_{01}||S_{12}| \ll a^{-1}\ln(a^{-1}), \tag{126}$$

which holds, provided $\|\phi\| \ll 1$. Therefore g is invertible and its inverse is of the form $g^{-1} = \frac{a}{\ln(a^{-1})}(1-o(1))$. Hence the equations (119) can be rewritten as (49) – (50), with $S_i(\phi,a,b,c) = \frac{f_i}{g}$ and $\mathcal{R}_i(\phi,a,b,c) = \frac{r_i}{g}$. Then the estimates of f_i, r_i and g given above, imply (51) – (52).

Proof of estimates (110) **-** (113)**.** Use $e^{-ay^2/2} = -\frac{1}{ay}\partial_y e^{-ay^2/2}$ and integration by parts to obtain

$$\langle 1, \zeta_{bc0}\rangle = \frac{1}{16}\int_0^\infty y(c+y^2)^{2d}e^{-a\frac{y^2}{2}}dy \tag{127}$$

$$= \frac{1}{16a}(c^{2d} + 4d\int_0^\infty (c+y^2)^{2d-1}ye^{-a\frac{y^2}{2}}dy)\,. \tag{128}$$

To extract the leading part in the last integral above we rescale $y \to \sqrt{a}y$ to obtain

$$\int_0^\infty (c+y^2)^{2d-1}ye^{-a\frac{y^2}{2}}dy = a^{-2d}\int_0^\infty (ac+y^2)^{2d-1}ye^{-\frac{y^2}{2}}dy\,.$$

Next, split the integral up into

$$\int_0^\infty (ac+y^2)^{2d-1}ye^{-\frac{y^2}{2}}dy = \int_0^1 (ac+y^2)^{2d-1}ye^{-\frac{y^2}{2}}dy + \int_1^\infty (ac+y^2)^{2d-1}ye^{-\frac{y^2}{2}}dy\,.$$

The second term on the left hand side is uniformly bounded in a, d small, so it suffices to investigate the first term. Write

$$\int_0^1 (ac+y^2)^{2d-1} y\mathrm{e}^{-\frac{y^2}{2}}\mathrm{d}y = \int_0^1 (ac+y^2)^{2d-1} y\mathrm{d}y + \int_0^1 (ac+y^2)^{2d-1} y(\mathrm{e}^{-\frac{y^2}{2}}-1)\mathrm{d}y\,,$$

where again the second term is uniformly bounded in a, d small. Explicit integration in the first term yields

$$\int_0^1 (ac+y^2)^{2d-1} y\mathrm{d}y = \frac{1}{4d}((1+ac)^{2d}-(ac)^{2d})\,.$$

By assumption, there is an $\varepsilon > 0$c such that $d(a) \leq a^{\varepsilon}$. In particular, $a^d \to 1$ as $a \to 0$ so

$$(1+ac)^{2d}-(ac)^{2d} = (1+2dO(ac))-(1+2dO(\ln ac)) = O(d\ln\frac{1}{a})\,,$$

yielding

$$\langle 1, \zeta_{bc0}\rangle = \frac{1}{16a}(1+O(d\ln\frac{1}{a}))\,.$$

The remaining terms are estimated similarly. □

References

[1] M. Alber, N. Chen, T. Glimm, and P.M. Lushnikov. Phys. Rev. E. **73**, 051901 (2006).

[2] M. Alber, N. Chen, P. M. Lushnikov, and S. A. Newman. Physical Review Letters, **99**, 168102 (2007).

[3] W. Beckner, Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality, Ann. of Math., 2, 138 (1993), pp. 213242.

[4] A. L. Bertozzi, J. A. Carillo, Th. Laurent, Blow-up in multidimensional aggregation equations with mildly singular interaction kernels. Nonlinearity 22 (2009) 683 - 710.

[5] P. Biler. Growth and accretion of mass in an astrophysical model. *Appl. Math. (Warsaw)*, 23:179-189, 1995

[6] P. Biler. Local and global solvability of some parabolic systems modeling chemotaxis. *Adv. Math. Sci. Appl.*, 8:715-743, 1998

[7] P. Biler, G. Karch, and P. Laurencot. The 8π-problem for radially symmetric solutions of a chemotaxis model in the plane. *Preprint*, 2006

[8] P. Bizoń, Yu. N. Ovchinnikov, I. M. Sigal, Collapse of an instanton. Nonlinearity **17** (2004), no. 4, 1179 - 1191.

[9] A. Blanchet, J. Dolbeault, and B. Perthame. Two-dimensional Keller-Segel model: Optimal critical mass and qualitative properties of the solutions. *Electr. J. Diff. Eqns.*, 44:1-33, 2006

[10] A. Blanchet, E. Carlen, J. A. Carrillo, Functional inequalities, thick tails and asymptotics for the critical mass Patlak-Keller-Segel model. arXiv:1009.0134.

[11] A. Blanchet, J. A. Carrillo, P. Laurençot, Critical mass for a Patlak-Keller-Segel model with degenerate diffusion in higher dimensions. Calc. Var. Partial Differential Equations 35 (2009), no. 2, 133 - 168.

[12] A. Blanchet, J. A. Carrillo, N. Masmoudi, Infinite time aggregation for the critical Patlak-Keller-Segel model in $\mathbb{R}^2$. Comm. Pure Appl. Math. 61 (2008), no. 10, 1449 - 1481.

[13] A. Blanchet, J. Dolbeault, M. Escobedo, J. Fernandez, Asymptotic behaviour for small mass in the two-dimensional parabolic-elliptic Keller-Segel model. J. Math. Anal. Appl. 361 (2010), no. 2, 533 - 542.

[14] A. Blanchet, J. Dolbeault, B. Perthame, Two-dimensional Keller-Segel model: optimal critical mass and qualitative properties of the solutions. Electron. J. Differential Equations, No. 44:1-33, 2006.

[15] J.T. Bonner. The cellular slime molds. *Princeton University Press*, Princeton, New Jersey, second edition, 1967

[16] M.P. Brenner, P. Constantin, L.P. Kadanoff, A. Schenkel, S.C. Venkataramani. Diffusion, attraction and collapse. *Nonlinearity*, 12:1071-1098, 1999

[17] M.P. Brenner, L.S. Levitov, and E.O. Budrene. Physical mechanisms for chemotactic pattern formation by bacteria. *Biophys. J.*, 74:1677-1693, 1998

[18] Buslaev, V.S. and Perel'man, G.S., On the stability of solitary waves for nonlinear Schrödinger equations. *Amer. Math. Soc. Transl. Ser.*, **2**, 74–98 (1995).

[19] Buslaev, V.S. and Sulem C., On the stability of solitary waves for nonlinear Schrödinger equations, *Ann. IHP. Analyse Nonlineéaire*, **20**, 419–475 (2003).

[20] E. Carlen and M. Loss, Competing symmetries, the logarithmic HLS inequality and Onofris inequality on Sn, Geom. Funct. Anal., 2 (1992), pp. 90104.

[21] E. Carlen and A. Figalli. Stability for a GNS inequality and the Log-HLS inequality, with application to the critical mass Keller-Segel equation. *Preprint*, arXiv:1107.5976

[22] P. Carmeliet, Mechanisms of angiogenesis and arteriogenesis. Nat. Med. **6**, 389-395 (2000).

[23] Carrillo, Jos A.; Fornasier, Massimo; Toscani, Giuseppe; Vecil, Francesco Particle, kinetic, and hydrodynamic models of swarming. Mathematical modeling of collective behavior in socio-economic and life sciences, 297 - 336, Model. Simul. Sci. Eng. Technol., Birkhäuser Boston, Inc., Boston, MA, 2010.

[24] P.-H. Chavanis, C. Sire, Exact analytical solution of the collapse of self-gravitating Brownian particles and bacterial populations at zero temperature. Phys. Rev. E (3) 83 (2011), 031131

[25] S. Childress and J.K. Percuss. Nonlinear aspects of chemotaxis. *Math. Bisosc.*, 56:217-237, 1981.

[26] Constantin, P.; Kevrekidis, I. G.; Titi, E. S. Asymptotic states of a Smoluchowski equation. Arch. Ration. Mech. Anal. 174 (2004), no. 3, 365384.

[27] Constantin, Peter; Kevrekidis, Ioannis; Titi, Edriss S. Remarks on a Smoluchowski equation. Discrete Contin. Dyn. Syst. 11 (2004), no. 1, 101112.

[28] H.L. Cycon, R.G. Froese , W. Kirsch and B. Simon. Schrödinger Operators with application to quantum mechanics and global geometry. *Texts and Monographs in Physics. Springer study edition. Springer-Verlag Berlin* (1987).

[29] S. Dejak, Z. Gang, I. M. Sigal, S. Wang. Blowup dynamics in nonlinear heat equations. Adv Appl Math 40, 433, 2008.

[30] S.I. Dejak, P.M. Lushnikov, Yu.N. Ovchinnikov, and I.M. Sigal On spectra of linearized operators for Keller-Segel models of chemotaxis *Physica D*, 241:1245-1254, 2012.

[31] Doi, M.: Molecular dynamics and rheological properties of concentrated solutions of rodlike polymers in isotropic and liquid crystalline phases. J. Polym. Sci., Polym. Phys. Ed. 19, 229243 (1981).

[32] S.A. Dyachenko, P.M. Lushnikov and N. Vladimirova. *Logarithmic-type Scaling of the Collapse of Keller-Segel Equation.* AIP Conf. Proc. **1389**, 709-712 (2011).

[33] H. Gajewski and K. Zacharias. Global behavior of a reaction-diffusion system modelling chemotaxis. *Math. Nachr.*, 195:77-114, 1998

[34] Gang Zhou and Sigal, I.M.. On Soliton Dynamics in Nonlinear Schrödinger Equations, *GAFA*, **16** (2006) 1377-1390.

[35] Gang Zhou and Sigal, I.M.. Relaxation of solitons in nonlinear Schrödinger equations with potentials, *Adv Math*, 216 (2007), no. 2, 443–490.

[36] S. Gustafson and I.M. Sigal. Mathematical Concepts of Quantum Mechanics. *Springer-Verlag*, Berlin, Heidelberg, 2011

[37] M.A. Herrero, and J.J.L. Velázquez. Chemptactic collapse for the Keller-Segel model. *J. Math. Biol.*, 35:177-194, 1996

[38] M.A. Herrero, and J.J.L. Velázquez. Singularity patterns in a chemotaxis model. *Math. Ann.*, 306:583-623, 1996

[39] M.A. Herrero, and J.J.L. Velázquez. A blowup mechanism for a chemotaxis model. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, 24:633-683, 1997

[40] M.A. Herrero, E. Medina, and J.J.L. Velázquez. Finite-time aggregation into a single point in a reaction-diffusion system. *Nonlinearity*, 10:1739-1754, 1997

[41] M.A. Herrero, E. Medina, and J.J.L. Velázquez. Self-similar blowup for a reaction-diffusion system. *J. Comput. Appl. Math.*, 97:99-119, 1998

[42] T. Hillen and K. J. Painter, A user's guide to PDE models for chemotaxis, Journal of Mathematical Biology, 58 (2009), pp. 183217.

[43] D. Horstmann. The nonsymmetric case of the Keller-Segel model in chemotaxis: some recent results. *NoDEA Nonlinear Differential Equations Appl.*, 8:399-423, 2001

[44] D. Horstmann. On the existence of radially symmetric blowup solutions for the Keller-Segel model. *J. Math. Biol.*, 44:463-478, 2002

[45] D. Horstmann. From 1970 until present: The Keller-Segel model in chemotaxis and its consequences. I. *Jahresber. Deutsch. Math.-Verein*, 105:103-165, 2003

[46] D. Horstmann. From 1970 until present: The Keller-Segel model in chemotaxis and its consequences. II. *Jahresber. Deutsch. Math.-Verein*, 106:51-69, 2004

[47] D. Horstmann and G. Wang. Blowup in a chemotaxis model without symmetry assumptions. *European J. Appl. Math.*, 12:159-177, 2001

[48] D. Horstmann and M. Winkler. Boundedness vs. blowup in a chemotaxis system. *J. Differential Equations*, 215:52-107, 2005

[49] W. Jäger and S. Luckhaus. On explosions of solutions to a system of partial differential equations modelling chemotaxis. *Trans. Amer. Soc.*, 329:819-824, 1992

[50] E.F. Keller and L.A. Segel. Initiation of slime mold aggregation viewed as an instability. *J. Theor. Biol.*, 26:300-415, 1970

[51] J. Krieger, W. Schlag and D. Tataru, Renormalization and blow up for charge one equivariant critical wave maps, Invent. Math. 171 (2008), no. 3, 543 - 615.

[52] J. Krieger, W. Schlag and D. Tataru, Renormalization and blow up for critical Yang-Mills problem, e-print, arXiv:0809.211, 2008.

[53] Larson, R.G.: The Structure and Rheology of Complex Fluids. Oxford University Press, London, 1999

[54] P.M. Lushnikov. *Critical chemotactic collapse.* Physics Letters A, **374**, 1678-1685 (2010).

[55] P. M. Lushnikov, N. Chen, and M. Alber. Physical Review E, **78**, 061904 (2008).

[56] F. Merle and P. Rafael. Blow up of the critical norm for some radial $L2$ super critical nonlinear Schrodinger equations. Amer. J. Math. 130, 945(2008).

[57] F. Merle and H. Zaag, Blow-up behavior outside the origin for a semilinear wave equation in the radial case. Bull. Sci. Math. 135 (2011), no. 4, 353 - 373

[58] V. Nanjundiah. Chemotaxis, signal relaying and aggregation morphology. *J. Theor. Biol.*, 42:63-105, 1973

[59] T. Nagai. Blow-up of radially symmetric solutions to a chemotaxis system. *Adv. Math. Sci. Appl.*, 5:581-601, 1995

[60] T. Nagai. Blow-up of nonradial solutions to parabolic-elliptic systems modeling chemotaxis. *J. Inequal. Appl.*, 6:37-55, 2001

[61] T. Nagai. Global existence and blowup of solutions to a chemotaxis system. *Proceedings of the Third World Congress of Nonlinear Analysis, Part2 (Catania, 2000)*, 47:777-787, 2001

[62] T. Nagai and T. Senba. Behavior of radially symmetric solutions of a system related to chemotaxis. *Proceedings of the Third World Congress of Nonlinear Analysis, Part 6 (Athens, 1996)*, 30:3837-3842, 1997

[63] T. Nagai and T. Senba. Global existence and blowup of radial solutions to a parabolic-elliptic system of chemotaxis. *Adv. Math. Sci. Appl.*, 8:145-156, 1998

[64] T. Nagai, T. Senba, and T. Suzuki. Chemotactic collapse in a parabolic system of mathematical biology. *Hiroshima Math. J.*, 30:463-497, 2000

[65] T. Nagai, T. Senba, and K. Yoshida. Application of the Trudinger-Moser inequality to a parabolic system of chemotaxis. *Funkcial. Ekvac.*, 40:411-433, 1997

[66] T. J. Newman, and R. Grima. Phys. Rev. E **70**, 051916 (2004).

[67] K. Oelschläger. On the derivation of reaction-diffusion equations as limit dynamics of systems of moderately interacting stochastic processes. *Probab. Theory Related Fields*, 82:565-586, 1989

[68] J. Ostriker, Ap J 140 10560, 1964.

[69] H.G. Othmer and A. Stevens. Aggregation, blowup, and collapse: the ABCs of taxis in reinforced random walks. *SIAM J. Appl. Math.*, 57:1044-1081, 1997

[70] Yu. N. Ovchinnikov, I. M. Sigal, On Collapse of Wave Maps. Physica D 240 (2011), pp. 1311 - 1324.

[71] C. S. Patlak. Random walk with persistence and external bias. Bull. Math. Biophys **15**, 311 (1953).

[72] B. Perthame, Transport equations in biology, Frontiers in Mathematics, Birkhauser Verlag, Basel, 2007.

[73] P.Rafael and I.Rodnianski. Stable blow up dynamics for the critical co-rotational Wave Maps and equivariant Yang-Mills problems. arXiv:0911.0692, 2010.

[74] M. Reed and B. Simon. Methods of Modern Mathematical Physics IV. *Academic Press*, San Diego, California, 1979

[75] I. Rodnianski and J.Sternbenz, On the formation of singularities in the critical $O(3)$ σ-model, Ann. of Math. (2) 172 (2010), no. 1, 187 - 242.

[76] C. Sire. P.-H. Chavanis, Critical dynamics of self-gravitating Langevin particles and bacterial populations. Phys. Rev. E (3) 78 (2008), 061111

[77] Soffer, A. and Weinstein, I.M. Multichannel nonlinear scattering for nonintegrable equations. *Comm. Math. Phys.*, **133**, 119–146 (1990).

[78] Soffer, A. and Weinstein, I.M. Multichannel nonlinear scattering for nonintegrable equations II. The case of anisotropic potentials and data. *J. Diff. Equations*, **98**, 376–390 (1992).

[79] Soffer, A. and Weinstein, I.M., Selection of the ground state for nonlinear Schrödinger equations. Rev. Math. Phys, ArXiv:nlin.PS/0308020 (2003).

[80] A. Stevens. The derivation of chemotaxis equations as limit dynamics of moderately interacting stochastic many-particle systems. *SIAM J. Appl. Math.*, 61:183-212 (electronic), 2000

[81] M. Struwe, Equivariant wave maps in two space dimensions. Comm. Pure Appl. Math. 56 (2003), 815-823.

[82] C. Sulem and P.L. Sulem, Nonlinear Schrödinger Equation. *Series in Mathematical Sciences, Volume 139, Springer-Verlag*, 1999.

[83] Tsai, T.-P. and Yau, H.-T., Asymptotic dynamics of nonlinear Schrödinger equations: resonance-dominated and dispersion-dominated solutions. *Comm. Pure Appl. Math.*, **55**, 153–216 (2002).

[84] Tsai, T.-P. and Yau, H.-T., Relaxation of excited states in nonlinear Schrödinger Equations, *Int. Math. Res. Not.*, **31**, 1629–1673 (2002).

[85] Tsai, T.-P. and Yau, H.-T., Stable directions for excited states of nonlinear Schrödinger equations, *Comm. PDE*, **27**, 2363–2402 (2002).

[86] J. J. L. Velázquez, Stability of some mechanisms of chemotactic aggregation. *SIAM J. Appl. Math.*, 62(5):1581 - 1633 (electronic), 2002.

[87] J. J. L. Velázquez, *SIAM J. Appl. Math.*, 64(4):1198 - 1248, 2004.

[88] G. Wolansky. On steady distributions of self-attracting clusters under friction and fluctuations. *Arch. Rational Mech. Anal.*, 119:355-391, 1992